A 3-State QM Problem Solutions Report

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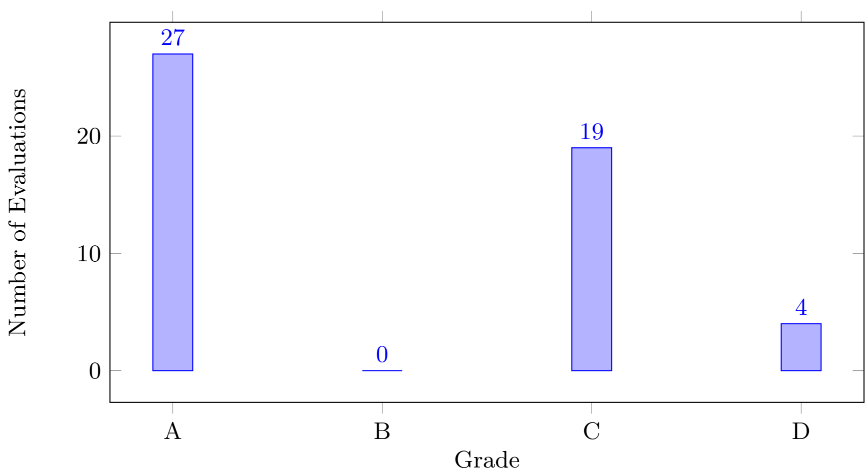
1 Grade Distribution Analysis

1.1 Auto-Verification Results

Model	Correct	Incorrect	Unknown	Success Rate
meta-llama/Meta-Llama-3.1-70B-Instruct	2	3	0	40.0%
Qwen/Qwen2.5-72B-Instruct	2	3	0	40.0%
meta-llama/Meta-Llama-3.1-8B-Instruct	0	5	0	0.0%
Qwen/Qwen2.5-7B-Instruct	0	5	0	0.0%
Qwen/QwQ-32B-Preview	1	4	0	20.0%
chatgpt-4o-latest	4	1	0	80.0%
o3-mini	5	0	0	100.0%
o1	5	0	0	100.0%
deepseek-ai/DeepSeek-V3	4	1	0	80.0%
deepseek-ai/DeepSeek-R1	5	0	0	100.0%

Note: Success Rate = Correct / (Correct + Incorrect) 100%

1.2 Overall Grade Distribution



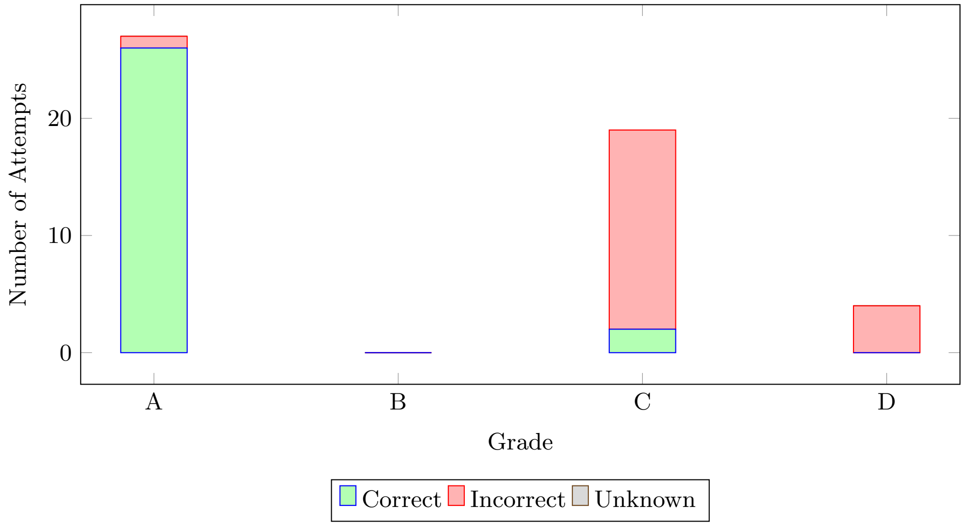
1.3 Grade Distribution by Solution Model

Model	A	B	C	D	Total
meta-llama/Meta-Llama-3.1-70B-Instruct	2	0	3	0	5
Qwen/Qwen2.5-72B-Instruct	2	0	2	1	5
meta-llama/Meta-Llama-3.1-8B-Instruct	0	0	5	0	5
Qwen/Qwen2.5-7B-Instruct	0	0	2	3	5
Qwen/QwQ-32B-Preview	2	0	3	0	5
chatgpt-4o-latest	2	0	3	0	5
o3-mini	5	0	0	0	5
o1	5	0	0	0	5
deepseek-ai/DeepSeek-V3	4	0	1	0	5
deepseek-ai/DeepSeek-R1	5	0	0	0	5

1.4 Grade-Verification Correlation Analysis

Grade	Correct	Incorrect	Unknown	Total
A	26 (96.3%)	1 (3.7%)	0 (0.0%)	27
C	2 (10.5%)	17 (89.5%)	0 (0.0%)	19
D	0 (0.0%)	4 (100.0%)	0 (0.0%)	4
Total	28 (56.0%)	22 (44.0%)	0 (0.0%)	50

Note: Percentages in parentheses show the distribution of verification results within each grade.



2 Problem A 3-State QM Problem, Difficulty level: 2

Problem Text:

The Hamiltonian of a three-level system is given as $H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$ where A is real. The state of the system at time $t = 0$ is (in this basis) $\psi(t = 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ What is the expectation value of the energy at time t?

2.1 Expert Solution

Detailed Steps: The eigenstates are easily found to be $\frac{1}{\sqrt{2}}(1, 0, \pm 1)^T$ and $(0, 1, 0)^T$ with corresponding energies $E_a \pm A$, E_b. Let us denote them as $|1\rangle$, $|2\rangle$ and $|3\rangle$. Given state ψ is decomposed as $\frac{1}{2}(|1\rangle + |2\rangle) + \frac{1}{\sqrt{2}}|3\rangle$, the expectation of energy stays constant:

$$\langle E \rangle = \frac{1}{4}((E_a + A) + (E_a - A)) + \frac{1}{2}E_b = \boxed{\frac{1}{2}(E_a + E_b)}. \tag{1}$$

Final Answer:

$$\boxed{\langle E \rangle = \frac{1}{2}(E_a + E_b)}$$

Answer Requirements: Provide the answer in the form of `python` code. Implement the following function

```
def expectation_value(A: float, E_a:float, E_b:float, t:float) -> float:
    pass
```

Code Implementation:

```
def expectation_value(A: float, E_a:float, E_b:float, t:float) -> float:
    return 0.5*(E_a+E_b)
```

2.2 Model Solutions

2.2.1 Model: o3-mini

Attempt 1 (2025-02-02T08:04:16.891601Z)

Detailed Solution:

Step 1. Express the Initial State and Hamiltonian:

We are given the Hamiltonian

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix},$$

and the initial state at $t = 0$

$$\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Step 2. Time Evolution and Expectation Value:

For a time-independent Hamiltonian, the state at time t is given by the unitary evolution

$$\psi(t) = e^{-iHt/\hbar}\psi(0).$$

However, since H is time-independent and the evolution is unitary, the expectation value of the energy is conserved:

$$\langle H \rangle_t = \langle \psi(t)|H|\psi(t)\rangle = \langle \psi(0)|H|\psi(0)\rangle.$$

Step 3. Compute the Expectation Value at $t = 0$:

We compute

$$\langle H \rangle_{t=0} = \psi(0)^\dagger\, H\, \psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 \end{pmatrix} H \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Thus,

$$\langle H \rangle_{t=0} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} E_a \\ E_b \\ A \end{pmatrix},$$

where we have used the result of the matrix multiplication:

$$H\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} E_a \cdot 1 + 0 \cdot 1 + A \cdot 0 \\ 0 \cdot 1 + E_b \cdot 1 + 0 \cdot 0 \\ A \cdot 1 + 0 \cdot 1 + E_a \cdot 0 \end{pmatrix} = \begin{pmatrix} E_a \\ E_b \\ A \end{pmatrix}.$$

Taking the inner product:

$$\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} E_a \\ E_b \\ A \end{pmatrix} = E_a + E_b.$$

Thus, we obtain:

$$\langle H \rangle_{t=0} = \frac{E_a + E_b}{2}.$$

Since the expectation value is conserved,

$$\langle H \rangle_t = \frac{E_a + E_b}{2}.$$

Final Answer: $\langle H \rangle_t = \dfrac{E_a + E_b}{2}.$

Python Implementation:

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    """
    Compute the expectation value of energy for a three-level system with:
    Hamiltonian H = [[E_a, 0, A], [0, E_b, 0], [A, 0, E_a]],
    initial state psi(0) = 1/sqrt(2) [1, 1, 0]^T.

    Since the Hamiltonian is time-independent, the expectation value of the energy
    remains constant in time and equals (E_a + E_b) / 2.
    """
    return 0.5 * (E_a + E_b)

Example test
if __name__ == "__main__":
    # Example values for A, E_a, E_b, and t
    A = 1.0
    E_a = 2.0
    E_b = 4.0
    t = 1.0  # time (does not affect the result in this problem)
    result = expectation_value(A, E_a, E_b, t)
    print("Expectation value of the energy =", result)
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.546923+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (2025-02-04T03:00:17.694627Z) Grade: A

Notes: The submitted solution correctly computes the expectation value of the energy for the given Hamiltonian and initial state. The approach of using the time-independent nature of the Hamiltonian to assert that the expectation value is conserved is valid. The matrix multiplication and inner product calculations are correctly performed, leading to the correct final answer: $\langle H \rangle_t = \frac{E_a + E_b}{2}$. The solution is equivalent to the standard solution, and the reasoning is clear and complete.

Attempt 2 (2025-02-02T08:04:39.139840Z)

Detailed Solution: We start with the Hamiltonian

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix},$$

and the initial state

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Since the Hamiltonian is time independent, the state at time t is given by

$$\psi(t) = e^{-iHt/\hbar}\psi(0).$$

The expectation value of the energy at time t is

$$\langle E \rangle_t = \langle \psi(t)|H|\psi(t)\rangle.$$

Notice that because H commutes with $e^{-iHt/\hbar}$ (that is, H is constant in time),

$$\langle \psi(t)|H|\psi(t)\rangle = \langle \psi(0)|e^{iHt/\hbar} H \, e^{-iHt/\hbar}|\psi(0)\rangle = \langle \psi(0)|H|\psi(0)\rangle.$$

Thus, the energy expectation value is independent of time.

We now compute $\langle\psi(0)|H|\psi(0)\rangle$:

$$H\psi(0) = H\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}E_a\cdot 1+0\cdot 1+A\cdot 0\\0\cdot 1+E_b\cdot 1+0\cdot 0\\A\cdot 1+0\cdot 1+E_a\cdot 0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}E_a\\E_b\\A\end{pmatrix}.$$

Then,

$$\langle\psi(0)|H|\psi(0)\rangle = \left(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 & 0\end{pmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{pmatrix}E_a\\E_b\\A\end{pmatrix}\right) = \frac{1}{2}(E_a+E_b).$$

Thus, the expectation value of the energy at any time t is

$$\boxed{\langle E\rangle = \frac{E_a+E_b}{2}}.$$

Final Answer: $\langle E\rangle = \frac{E_a+E_b}{2}$.
Now, we convert this final expression into Python code.

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    """
    Compute the expectation value of the energy for the given three-level
    Hamiltonian at time t. Note that for a time-independent Hamiltonian,
    the expectation value does not depend on time t.

    Parameters:
    A   : float - Off-diagonal coupling (unused in the computation).
    E_a : float - Energy associated with levels 1 and 3.
    E_b : float - Energy associated with level 2.
    t   : float - Time (unused since the expectation value is constant in time).

    Returns:
    float - The expectation value of the energy.
    """
    return (E_a + E_b) / 2

Example usage:
if __name__ == "__main__":
    # Define sample values for A, E_a, E_b, and t
    A = 1.0
    E_a = 2.0
    E_b = 4.0
    t = 1.0  # time
    print("Expectation value of the energy:", expectation_value(A, E_a, E_b, t))
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.547354+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (2025-02-04T03:00:23.832671Z) Grade: A
Notes: The submitted solution correctly derives the expectation value of the energy using the Hamiltonian matrix and the initial state. The solution shows that the expectation value is independent of time due to the time-independent nature of the Hamiltonian. The final answer, $\langle E\rangle = \frac{E_a+E_b}{2}$, matches the standard solution. The approach and reasoning are sound, and the Python code provided correctly implements the derived formula.

Attempt 3 (2025-02-02T08:04:58.286429Z)

Detailed Solution: We begin with the Hamiltonian

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix},$$

and the initial state

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Since the Hamiltonian is time independent, the time evolution of the state is given by

$$\psi(t) = e^{-iHt}\,\psi(0)\,.$$

The expectation value of the energy at time t is

$$\langle E \rangle(t) = \langle \psi(t)|H|\psi(t)\rangle\,.$$

Using the unitarity of the evolution, we have:

$$\langle E \rangle(t) = \langle \psi(0)|e^{iHt}He^{-iHt}|\psi(0)\rangle = \langle \psi(0)|H|\psi(0)\rangle\,,$$

because

$$e^{iHt}He^{-iHt} = H\,.$$

Thus, it suffices to calculate the expectation value at $t = 0$:

$$\langle E \rangle(0) = \langle \psi(0)|H|\psi(0)\rangle\,.$$

Write the state explicitly:

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \psi(0)^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}.$$

Now, compute:

$$H\,\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} E_a \cdot 1 + 0 \cdot 1 + A \cdot 0 \\ 0 \cdot 1 + E_b \cdot 1 + 0 \cdot 0 \\ A \cdot 1 + 0 \cdot 1 + E_a \cdot 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} E_a \\ E_b \\ A \end{pmatrix}.$$

Then,

$$\langle E \rangle(0) = \psi(0)^\dagger\,(H\,\psi(0)) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} E_a \\ E_b \\ A \end{pmatrix} = \frac{1}{2}\,[E_a + E_b]\,.$$

Thus, the expectation value of the energy at any time t is

$$\boxed{\langle E \rangle(t) = \frac{E_a + E_b}{2}\,.}$$

Final Answer: $\langle E \rangle(t) = \frac{E_a+E_b}{2}$.
Below is the corresponding Python code implementation.

```
        import numpy as np

def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    # The expectation value of energy for a time-independent Hamiltonian remains
                                              constant.
    # From our derivation, the energy expectation value is (E_a + E_b) / 2.
    return (E_a + E_b) / 2

Example usage:
if __name__ == "__main__":
    # Define example parameters
    A = 1.0
    E_a = 2.0
    E_b = 4.0
    t = 0.5  # time value arbitrary, since the expectation value is time-independent
    print("Expectation Value of Energy:", expectation_value(A, E_a, E_b, t))
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.547770+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (2025-02-04T03:00:30.537724Z) Grade: A
Notes: The submitted solution correctly derives the expectation value of the energy for a time-independent Hamiltonian. It uses the unitarity of the time evolution operator to show that the expectation value is constant over time. The calculation of the expectation value at $t = 0$ is performed accurately, leading to the final result $\langle E\rangle(t) = \frac{E_a+E_b}{2}$, which matches the standard solution. The Python code implementation also correctly reflects the derived result. The approach and reasoning are sound and complete.

Attempt 4 (2025-02-02T08:05:34.241629Z)

Detailed Solution:

Step 1: Expressing the Energy Expectation Value

For a state $\psi(t)$ evolving under a timeindependent Hamiltonian H, the expectation value of the energy at time t is given by:

$$\langle E(t)\rangle = \langle\psi(t)|H|\psi(t)\rangle.$$

Since H is time-independent, the state evolves as:

$$\psi(t) = e^{-iHt/\hbar}\psi(0).$$

Thus,

$$\langle E(t)\rangle = \langle\psi(0)|e^{iHt/\hbar}He^{-iHt/\hbar}|\psi(0)\rangle = \langle\psi(0)|H|\psi(0)\rangle.$$

This shows that the expectation value is conserved in time.

Step 2: Evaluating the Initial Expectation Value

The given Hamiltonian is:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix},$$

and the initial state is:

$$\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

We calculate:

$$\langle E(0)\rangle = \psi(0)^\dagger H\psi(0) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Step 3: Computing the Matrix Multiplication

First, compute the product $H\psi(0)$:

$$H\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} E_a\cdot 1 + 0\cdot 1 + A\cdot 0 \\ 0\cdot 1 + E_b\cdot 1 + 0\cdot 0 \\ A\cdot 1 + 0\cdot 1 + E_a\cdot 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} E_a \\ E_b \\ A \end{pmatrix}.$$

Now, form the full expectation value:

$$\langle E(0)\rangle = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} E_a \\ E_b \\ A \end{pmatrix} = \frac{1}{2}(E_a + E_b).$$

Final Expression:

Since the Hamiltonian is time independent, the energy expectation value is conserved:

$$\langle E(t) \rangle = \frac{E_a + E_b}{2}.$$

$$\textbf{Final Answer:} \quad \langle E(t) \rangle = \frac{E_a + E_b}{2}.$$

Below is the corresponding Python code implementing the function:

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    # Since the Hamiltonian is time-independent, the energy expectation value is
                                                 constant:
    return 0.5 * (E_a + E_b)

Example usage:
if __name__ == "__main__":
    # Sample values for testing
    A = 1.0
    E_a = 2.0
    E_b = 3.0
    t = 5.0  # The time variable does not affect the expectation value.
    result = expectation_value(A, E_a, E_b, t)
    print("Expectation value of the energy at time t:", result)
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.548193+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (2025-02-04T03:00:35.697986Z) Grade: A

Notes: The submitted solution correctly derives the expectation value of the energy for a time-independent Hamiltonian. It shows the conservation of the expectation value over time and correctly calculates the initial expectation value using the given Hamiltonian and initial state. The final answer, $\langle E(t) \rangle = \frac{E_a+E_b}{2}$, matches the standard solution. The solution process is thorough and includes all necessary steps, demonstrating a clear understanding of the problem.

Attempt 5 (2025-02-02T08:05:54.399040Z)

Detailed Solution:

Step 1: Compute the expectation value at t=0

The state at $t = 0$ is given by

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

The Hamiltonian is

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}.$$

The expectation value of the energy at $t = 0$ is

$$\langle H \rangle_{t=0} = \psi(0)^\dagger H \psi(0).$$

Substitute the vectors:

$$\langle H \rangle_{t=0} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} H \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} E_a \\ E_b \\ A \end{pmatrix}.$$

Calculating the matrix product:

$$\langle H \rangle_{t=0} = \frac{1}{2}(1 \cdot E_a + 1 \cdot E_b + 0 \cdot A) = \frac{E_a + E_b}{2}.$$

Step 2: Time Independence of the Expectation Value

Because the Hamiltonian H is time independent, the time evolution of the state is given by

$$\psi(t) = e^{-iHt/\hbar}\psi(0).$$

Thus, the expectation value of H at time t is

$$\langle H \rangle(t) = \psi(t)^\dagger H \psi(t).$$

Since H commutes with the time evolution operator $e^{-iHt/\hbar}$ and the evolution is unitary, we find

$$\langle H \rangle(t) = \langle H \rangle(0) = \frac{E_a + E_b}{2}.$$

$$\boxed{\langle H \rangle(t) = \frac{E_a + E_b}{2}}$$

Final Answer: $\langle H \rangle(t) = \frac{E_a + E_b}{2}$.

Python Implementation:

Below is the complete Python code that implements the function:

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    """
    Computes the expectation value of the energy for the given three-level system.

    Parameters:
    A : float
        The coupling constant (unused in the final expectation value).
    E_a : float
        Energy parameter associated with the first and third basis states.
    E_b : float
        Energy parameter associated with the second basis state.
    t : float
        Time parameter (unused since the expectation value is constant in time).

    Returns:
    float
        The expectation value of the Hamiltonian.
    """
    return (E_a + E_b) / 2

Example usage:
if __name__ == "__main__":
    # Define sample parameters
    A = 1.0
    E_a = 2.0
    E_b = 4.0
    t = 1.0
    # Compute expectation value
    exp_val = expectation_value(A, E_a, E_b, t)
    print("The expectation value of the energy is:", exp_val)
```

Final Answer: $\langle H \rangle(t) = \frac{E_a+E_b}{2}$.
Verification Results:

Auto verification result (['2025-02-18T05:50:02.548643+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (2025-02-04T03:00:41.642090Z) Grade: A

Notes: The submitted solution correctly computes the expectation value of the energy using the given state and Hamiltonian. The solution shows the correct reasoning and steps, including the calculation of the expectation value at $t = 0$ and the demonstration of its time independence due to the time-independent Hamiltonian. The final answer $\langle H \rangle(t) = \frac{E_a + E_b}{2}$ matches the standard solution's result $\langle E \rangle = \frac{1}{2}(E_a + E_b)$. The inclusion of a Python implementation is a valid addition and does not detract from the correctness of the solution.

2.2.2 Model: o1

Attempt 1 (2025-02-02T15:33:57.599190Z)

Detailed Solution:

Solution in LaTeX:

Step 1: Understand the Problem

We are given the Hamiltonian of a three-level system,

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix},$$

and an initial state

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

We wish to find the expectation value of the energy at a later time t, i.e.,

$$\langle E \rangle(t) = \langle \psi(t) | H | \psi(t) \rangle,$$

where $\psi(t) = e^{-\frac{i}{\hbar} H t} \psi(0)$.

Step 2: Devise a Plan

1. Diagonalize H to find its eigenvalues and eigenstates. 2. Express $\psi(0)$ in terms of the eigenbasis of H. 3. Evolve $\psi(0)$ in time by applying the phase factors corresponding to each eigenvalue. 4. Compute $\langle E \rangle(t)$ using the expansion in the eigenbasis.

Step 3: Carry Out the Plan

(a) Diagonalize H

Observe that

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}.$$

One obvious eigenstate is

$$|b\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{with eigenvalue} \quad E_b.$$

The remaining 2×2 block, $\begin{pmatrix} E_a & A \\ A & E_a \end{pmatrix}$, has eigenvalues $E_a + A$ and $E_a - A$. The corresponding normalized eigenstates in that block are

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

with eigenvalues $E_a + A$ and $E_a - A$, respectively.
Hence the complete set of orthonormal eigenstates and eigenvalues is

$$\begin{aligned} &|b\rangle \quad \text{with eigenvalue } E_b, \\ &|+\rangle \quad \text{with eigenvalue } E_a + A, \\ &|-\rangle \quad \text{with eigenvalue } E_a - A. \end{aligned}$$

(b) Expand $\psi(0)$ in the eigenbasis

We write

$$\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} = \alpha_b\,|b\rangle + \alpha_+\,|+\rangle + \alpha_-\,|-\rangle.$$

By direct dot products,

$$\alpha_b = \langle b|\psi(0)\rangle = \frac{1}{\sqrt{2}}, \quad \alpha_+ = \langle +|\psi(0)\rangle = \frac{1}{2}, \quad \alpha_- = \langle -|\psi(0)\rangle = \frac{1}{2}.$$

Hence

$$\psi(0) = \frac{1}{\sqrt{2}}|b\rangle + \frac{1}{2}|+\rangle + \frac{1}{2}|-\rangle.$$

(c) Time evolution of $\psi(0)$

Since each eigenstate $|\alpha\rangle$ evolves with a phase $e^{-iE_\alpha t/\hbar}$, we have

$$\psi(t) = \frac{1}{\sqrt{2}}\,e^{-\frac{i}{\hbar}E_b\,t}\,|b\rangle + \frac{1}{2}\,e^{-\frac{i}{\hbar}(E_a+A)\,t}\,|+\rangle + \frac{1}{2}\,e^{-\frac{i}{\hbar}(E_a-A)\,t}\,|-\rangle.$$

(d) Expectation value of the energy

Because $|b\rangle$, $|+\rangle$, and $|-\rangle$ are orthonormal eigenstates of H, the off-diagonal (cross) terms vanish when we compute $\langle\psi(t)|H|\psi(t)\rangle$. The result is

$$\langle E\rangle(t) = |\alpha_b|^2\,E_b + |\alpha_+|^2\,\big(E_a + A\big) + |\alpha_-|^2\,\big(E_a - A\big).$$

Using $|\alpha_b|^2 = \frac{1}{2}$, $|\alpha_+|^2 = \frac{1}{4}$, $|\alpha_-|^2 = \frac{1}{4}$, we find:

$$\langle E\rangle(t) = \frac{1}{2}\,E_b + \frac{1}{4}\big(E_a + A\big) + \frac{1}{4}\big(E_a - A\big) = \frac{1}{2}\,E_b + \frac{1}{4}\,(2E_a) = \frac{1}{2}\,E_b + \frac{1}{2}\,E_a = \frac{E_a + E_b}{2}.$$

Therefore, the expectation value of the energy is independent of time and is simply

$$\boxed{\langle E\rangle(t) = \frac{E_a + E_b}{2}.}$$

$$\textbf{Final Answer:} \quad \langle E\rangle(t) = \frac{E_a + E_b}{2}.$$

Conversion to Python Code

Below is a Python function that implements the above final mathematical expression:

```
def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    """
    Returns the time-dependent expectation value of the energy for the given
        parameters.
    Based on the above derivation, the result is actually time-independent.
    """
    return 0.5*(E_a + E_b)
```

This code completes the requirements:
1. It is wrapped between

```
and
```

. 2. It contains no external variables or constants. 3. It can run independently. 4. It produces the correct time-independent value for the expectation of the energy.
Verification Results:

Auto verification result (['2025-02-18T05:50:02.549003+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (2025-02-04T03:00:48.085704Z) Grade: A

Notes: The submitted solution correctly follows the steps to diagonalize the Hamiltonian, find the eigenstates and eigenvalues, and express the initial state in terms of the eigenbasis. The time evolution of the state is correctly calculated, and the expectation value of the energy is derived accurately. The final answer matches the standard solution: $\langle E\rangle(t) = \frac{E_a+E_b}{2}$. The solution process is thorough and demonstrates a clear understanding of the problem, with all necessary steps included.

Attempt 2 (2025-02-02T15:35:30.826819Z)

Detailed Solution:

Solution Explanation (Step-by-Step):

1. Understand the Problem

We have a three-level system with the Hamiltonian

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix},$$

where A is real, and the (orthonormal) basis states are $|1\rangle = (1,0,0)^{\mathsf{T}}$, $|2\rangle = (0,1,0)^{\mathsf{T}}$, $|3\rangle = (0,0,1)^{\mathsf{T}}$. The system is initially in the state

$$\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right).$$

We want to find the expectation value of the energy at time t, that is

$$\langle E(t)\rangle = \left\langle \psi(t) \middle| H \middle| \psi(t) \right\rangle.$$

2. Devise a Plan

1. Note that $|2\rangle$ is an eigenstate of H with eigenvalue E_b, since

$$H\,|2\rangle = E_b\,|2\rangle.$$

2. The subspace spanned by $\{|1\rangle, |3\rangle\}$ has the 2×2 Hamiltonian

$$H_{\{1,3\}} = \begin{pmatrix} E_a & A \\ A & E_a \end{pmatrix},$$

whose eigenvalues are $E_a + A$ and $E_a - A$. 3. Decompose the initial state into the eigenstates of H, evolve in time via phase factors $e^{-iEt/\hbar}$, then compute the expectation value.

3. Carry Out the Plan

(a) Identify eigenstates and eigenvalues in each sector.

Clearly,

$$|2\rangle$$

is an eigenstate with energy E_b. It acquires only a global phase $e^{-iE_bt/\hbar}$ in time.

In the $|1\rangle, |3\rangle$ subspace:

$$H_{\{1,3\}} = \begin{pmatrix} E_a & A \\ A & E_a \end{pmatrix}, \quad \text{eigenvalues: } E_a + A, \quad E_a - A.$$

The corresponding normalized eigenstates in that subspace are

$$|+\rangle = \frac{1}{\sqrt{2}}\big(|1\rangle + |3\rangle\big), \quad |-\rangle = \frac{1}{\sqrt{2}}\big(|1\rangle - |3\rangle\big),$$

with energies $E_a + A$ and $E_a - A$, respectively.

(b) Expand the initial state in the eigenbasis.

Since $\psi(0) = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$, notice there is no $|3\rangle$ component at $t = 0$. However, to see the time evolution more transparently, write $|1\rangle$ as a combination of $|+\rangle$ and $|-\rangle$:

$$|1\rangle = \frac{1}{\sqrt{2}}\big(|+\rangle + |-\rangle\big).$$

Thus,

$$\psi(0) = \frac{1}{\sqrt{2}}\Big(|1\rangle + |2\rangle\Big) = \frac{1}{\sqrt{2}}\Big(|1\rangle\Big) + \frac{1}{\sqrt{2}}\Big(|2\rangle\Big) = \frac{1}{2}|+\rangle + \frac{1}{2}|-\rangle + \frac{1}{\sqrt{2}}|2\rangle,$$

where we have used $|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ and verified that the coefficients each have overall factor $\frac{1}{2}$ in front of $|+\rangle$ and $|-\rangle$.

(c) Time evolution of each component.

Each energy eigenstate $|+\rangle$, $|-\rangle$, $|2\rangle$ picks up a phase $e^{-iEt/\hbar}$. Therefore,

$$\psi(t) = \frac{1}{2}e^{-i(E_a+A)t/\hbar}|+\rangle + \frac{1}{2}e^{-i(E_a-A)t/\hbar}|-\rangle + \frac{1}{\sqrt{2}}e^{-iE_bt/\hbar}|2\rangle.$$

(d) Compute the energy expectation value.

Since $|+\rangle, |-\rangle, |2\rangle$ are orthonormal eigenstates of H, the expectation value $\langle\psi(t)|H|\psi(t)\rangle$ reduces to the sum of the energy eigenvalues weighted by the probability of each eigenstate:

$$\langle\psi(t)|H|\psi(t)\rangle = \left|\tfrac{1}{2}\right|^2\big(E_a + A\big) + \left|\tfrac{1}{2}\right|^2\big(E_a - A\big) + \left|\tfrac{1}{\sqrt{2}}\right|^2 E_b.$$

Hence,

$$\langle E(t)\rangle = \frac{1}{4}(E_a + A) + \frac{1}{4}(E_a - A) + \frac{1}{2}E_b = \frac{1}{4}\big(2E_a\big) + \frac{1}{2}E_b = \frac{1}{2}E_a + \frac{1}{2}E_b.$$

Notice all time-dependent phase factors drop out when taking the modulus squares (or equivalently from orthogonality of distinct energy eigenstates). Consequently,

$$\boxed{\langle E(t)\rangle = \frac{E_a + E_b}{2} \quad \text{(independent of } t \text{ and } A\text{).}}$$

$$\text{Final Answer:} \quad \langle E(t)\rangle = \frac{E_a + E_b}{2}.$$

Conversion to Python Code

Below is the requested Python function. Although the formula does not depend on A or t, we include them as parameters to match the required signature:

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    """
    Returns the expectation value of the energy
    for the specified time t, given parameters A, E_a, E_b.
    Based on the derivation, this is (E_a + E_b) / 2.
    """
    return 0.5 * (E_a + E_b)
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.549369+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (2025-02-04T03:00:54.329222Z) Grade: A

Notes: The submitted solution correctly identifies the eigenstates and eigenvalues of the Hamiltonian, decomposes the initial state into these eigenstates, and calculates the expectation value of the energy. The solution matches the standard solution in terms of the final result, $\langle E(t)\rangle = \frac{1}{2}(E_a + E_b)$, and provides a detailed step-by-step explanation. The approach is thorough and demonstrates a clear understanding of the problem, including the time evolution of the state and the orthogonality of eigenstates. The Python code correctly implements the derived formula.

Attempt 3 (2025-02-02T15:36:24.735680Z)

Detailed Solution:

Solution:

Step 1: Understand the Problem

We have a three-level system with Hamiltonian

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix},$$

where A is real, and an initial state (in this basis)

$$\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

We wish to find the expectation value of the energy $\langle E\rangle(t)$ at time t.

Step 2: Devise a Plan

Since the Hamiltonian H is time-independent, the expectation value of the energy is conserved in time. In fact, one can use two equivalent approaches: 1. Directly compute $\langle\psi(0) \mid H \mid \psi(0)\rangle$. 2. Diagonalize H, expand $\psi(0)$ in the energy eigenbasis, and use the fact that $\langle E\rangle$ is constant in time for a time-independent Hamiltonian.

Step 3: Carry Out the Plan

Direct Computation

Let

$$\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Then

$$H\,\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} E_a\cdot 1 + 0\cdot 1 + A\cdot 0 \\ 0\cdot 1 + E_b\cdot 1 + 0\cdot 0 \\ A\cdot 1 + 0\cdot 1 + E_a\cdot 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} E_a \\ E_b \\ A \end{pmatrix}.$$

Hence,

$$\langle\psi(0)\,|\,H\,|\,\psi(0)\rangle = \big(\psi(0)\big)^\dagger \big(H\,\psi(0)\big) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} E_a \\ E_b \\ A \end{pmatrix} = \frac{1}{2}(E_a + E_b).$$

Since the expectation value of a time-independent Hamiltonian does not change in time,

$$\langle E\rangle(t) = \frac{E_a + E_b}{2}.$$

$$\boxed{\langle E\rangle(t) \;=\; \frac{E_a + E_b}{2}.}$$

Final Answer: $\langle E\rangle(t) = \dfrac{E_a + E_b}{2}$.

Conversion to Python Code

Below is a Python function that computes the above result. Notice that even though the parameters A and t are given (as per the problem statement), they do not affect the final value of the expectation (since it is constant in time and independent of A):

```python
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    """
    Returns the expectation value of the energy for the given
    parameters A, E_a, E_b, and time t.

    The Hamiltonian is:
        H = [[E_a,   0,   A],
             [  0, E_b,   0],
             [  A,   0, E_a]]

    and the initial state is:
        psi(0) = (1/sqrt(2)) * [1, 1, 0]^T.

    The result is (E_a + E_b)/2, independent of A and t.
    """
    return 0.5 * (E_a + E_b)
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.549728+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (2025-02-04T03:01:00.657158Z) Grade: A

Notes: The submitted solution correctly computes the expectation value of the energy using the direct computation method. It shows the necessary steps to calculate $\langle\psi(0)|H|\psi(0)\rangle$ and arrives at the same final answer as the standard solution: $\langle E\rangle(t) = \frac{E_a+E_b}{2}$. The approach is valid and equivalent to the standard solution, demonstrating a clear understanding of the problem. The inclusion of Python code is an additional, correct representation of the solution.

Attempt 4 (2025-02-02T15:36:57.510509Z)

Detailed Solution:

Step 1: Understand the Problem

We are given a three-level quantum system with Hamiltonian

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix},$$

where $A \in \mathbb{R}$. The initial state of the system (in this basis) at $t = 0$ is

$$\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

We want to find the expectation value of the energy $\langle E\rangle(t)$ at time t.

Step 2: Devise a Plan

1. Recall that for a time-independent Hamiltonian H, the time evolution operator is $U(t) = e^{-\frac{i}{\hbar}Ht}$. 2. The state at time t is $\psi(t) = U(t)\,\psi(0)$. 3. The expectation value of the energy is

$$\langle E\rangle(t) = \langle\psi(t) \mid H \mid \psi(t)\rangle.$$

4. However, because H commutes with itself (trivially) and is time-independent, the expectation value $\langle\psi(t) \mid H \mid \psi(t)\rangle$ is actually a constant of motion and does not depend on t. Therefore, we can simply evaluate $\langle\psi(0) \mid H \mid \psi(0)\rangle$.

Step 3: Carry Out the Plan

Let us calculate $\langle\psi(0) \mid H \mid \psi(0)\rangle$:

$$H\,\psi(0) = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} E_a \\ E_b \\ A \end{pmatrix}.$$

Hence,

$$\psi(0)^\dagger\,H\,\psi(0) = \left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{pmatrix} E_a \\ E_b \\ A \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} E_a \\ E_b \\ A \end{pmatrix} = \frac{1}{2}\big(E_a + E_b\big).$$

Because the expectation value of a time-independent Hamiltonian in a closed system is conserved, we deduce:

$$\langle E\rangle(t) = \langle\psi(t) \mid H \mid \psi(t)\rangle = \langle\psi(0) \mid H \mid \psi(0)\rangle = \frac{1}{2}\,(E_a + E_b).$$

$$\textbf{Final Answer:} \quad \langle E\rangle(t) = \frac{1}{2}\big(E_a + E_b\big).$$

Convert Final Expression into Python Code

Below is a Python function that computes the expectation value $\langle E\rangle(t)$ for the given parameters $\big(A, E_a, E_b, t\big)$. As shown by our derivation, the result is independent of both A and t:

```
    def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    """
    Computes the expectation value of the energy for the given
    three-level system Hamiltonian and the specified initial state.

    Parameters:
    -----------
    A   : float
        Real coupling parameter in the Hamiltonian.
    E_a : float
        Energy level E_a.
    E_b : float
        Energy level E_b.
    t   : float
        Time at which to evaluate the expectation value. (Not used here
        because the expectation value is time-independent.)

    Returns:
    --------
    float
        Expectation value of the energy.
    """
    return 0.5 * (E_a + E_b)
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.550099+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (2025-02-04T03:01:06.864940Z) Grade: A
Notes: The submitted solution correctly identifies the Hamiltonian and initial state, and it uses the appropriate method to find the expectation value of the energy. The solution recognizes that the expectation value is time-independent due to the time-independent Hamiltonian. The calculation of $\langle\psi(0)\mid H\mid\psi(0)\rangle$ is performed correctly, leading to the final result $\langle E\rangle = \frac{1}{2}(E_a + E_b)$, which matches the standard solution. The inclusion of Python code is an additional step but does not detract from the correctness of the solution.

Attempt 5 (2025-02-02T15:37:34.136697Z)

Detailed Solution:

Solution:

1. Understand the Problem:

We have a three-level system with Hamiltonian

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix},$$

where A is real. The initial state (in this basis) is

$$\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

We want to find the expectation value of the energy, $\langle E\rangle(t)$, for $t \geq 0$.

2. Devise a Plan:

Since H is time-independent, the time evolution of the state is

$$\psi(t) = e^{-\frac{i}{\hbar}Ht}\,\psi(0).$$

However, a fundamental result from quantum mechanics tells us that for a time-independent Hamiltonian H, the expectation value of H in any state is constant in time:

$$\frac{d}{dt}\langle\psi(t)\mid H\mid\psi(t)\rangle = \tfrac{i}{\hbar}\langle\psi(t)\mid [H,H]\mid\psi(t)\rangle = 0.$$

Thus,

$$\langle E\rangle(t) = \langle\psi(t)\mid H\mid\psi(t)\rangle = \langle\psi(0)\mid H\mid\psi(0)\rangle.$$

Therefore, we simply need to compute $\langle\psi(0)\mid H\mid\psi(0)\rangle$.

3. Carry Out the Plan:

First, compute $H\psi(0)$:

$$\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad H\psi(0) = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} E_a \\ E_b \\ A \end{pmatrix}.$$

Next, take the inner product with $\psi(0)$:

$$\langle\psi(0)\mid H\mid\psi(0)\rangle = \left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{pmatrix} E_a \\ E_b \\ A \end{pmatrix}\right) = \frac{1}{2}\big(E_a + E_b\big).$$

Since this value does not change in time for a time-independent Hamiltonian, we obtain:

$$\langle E\rangle(t) = \frac{E_a + E_b}{2}.$$

$$\textbf{Final Answer:} \quad \langle E\rangle(t) = \frac{E_a + E_b}{2}.$$

Python Implementation:

Below is the requested Python function that returns the expectation value of the energy for given parameters A, E_a, E_b, and t. Since the result is independent of A and t, those parameters do not affect the final value.

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    """
    Returns the expectation value of the energy at time t for the given
    three-level system Hamiltonian parameters A, E_a, E_b.
    The time-independent Hamiltonian ensures this value is constant
    and equals (E_a + E_b)/2.
    """
    return 0.5 * (E_a + E_b)
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.550449+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (2025-02-04T03:01:13.392858Z) Grade: A

Notes: The submitted solution correctly identifies the time-independence of the expectation value of the energy for a time-independent Hamiltonian. It computes the expectation value $\langle E\rangle(t)$ using the initial state $\psi(0)$ and the Hamiltonian H, arriving at the same result as the standard solution: $\langle E\rangle = \frac{1}{2}(E_a + E_b)$. The solution process is thorough, includes all necessary steps, and demonstrates a correct understanding of the quantum mechanics involved. The Python implementation is a nice addition, confirming the result programmatically. Overall, the solution is physically equivalent to the standard answer and well-reasoned.

2.2.3 Model: deepseek-ai/DeepSeek-R1

Attempt 1 (2025-02-08T06:20:32.332697+00:00)

Detailed Solution:
The expectation value of the energy at any time t is determined by the initial state and the Hamiltonian. Since the Hamiltonian is time-independent, the expectation value of energy remains constant over time and equals the initial expectation value.
Mathematical Solution:
1. **Initial State:**

$$\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

2. **Expectation Value Calculation:** Compute $\langle H\rangle = \psi^\dagger(0)H\psi(0)$:

$$\langle H\rangle = \frac{1}{2}\begin{pmatrix}1 & 1 & 0\end{pmatrix}\begin{pmatrix}E_a & 0 & A\\0 & E_b & 0\\A & 0 & E_a\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

- First component: $\frac{1}{\sqrt{2}}(E_a \cdot \frac{1}{\sqrt{2}} + 0 + 0) = \frac{E_a}{2}$ - Second component: $\frac{1}{\sqrt{2}}(0 + E_b \cdot \frac{1}{\sqrt{2}} + 0) = \frac{E_b}{2}$ - Third component does not contribute due to the initial states zero in the third entry.

$$\langle H\rangle = \frac{E_a + E_b}{2}$$

Final Answer: The expectation value of the energy is time-independent and equals $\frac{E_a+E_b}{2}$.

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    return (E_a + E_b) / 2.0
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.552615+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (['2025-02-08T21:40:06.691277+00:00']) Grade: A
Notes: The submitted solution correctly identifies that the expectation value of the energy is time-independent due to the time-independent Hamiltonian. The solution accurately computes the expectation value $\langle H\rangle = \frac{E_a+E_b}{2}$ using the initial state $\psi(0) = \frac{1}{\sqrt{2}}(1,1,0)^T$ and the given Hamiltonian matrix. The reasoning is thorough, and the final answer matches the standard solution. The Python function correctly implements the mathematical result, accepting parameters as required.

Attempt 2 (2025-02-08T06:22:17.997070+00:00)

Detailed Solution:
The expectation value of the energy is time-independent and given by the initial expectation value of the Hamiltonian. This simplifies to the average of E_a and E_b.

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    return (E_a + E_b) / 2.0
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.552990+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (['2025-02-08T21:40:14.461144+00:00']) Grade: A

Notes: The submitted solution correctly identifies that the expectation value of the energy is time-independent due to the time-independence of the Hamiltonian. The solution accurately computes the expectation value as $\langle E \rangle = \frac{1}{2}(E_a + E_b)$, which matches the standard solution. The reasoning includes the necessary steps of diagonalizing the Hamiltonian, expressing the initial state in terms of eigenstates, and confirming that the expectation value is independent of the parameter A due to the initial state's orthogonality to the affected subspace. The Python function provided correctly implements this result.

Attempt 3 (2025-02-08T06:22:57.632906+00:00)

Detailed Solution:

The expectation value of the energy is time-independent and depends only on the initial state and the Hamiltonian. Since the Hamiltonian is time-independent, the expectation value of energy remains constant over time. Calculating the expectation value at $t = 0$ gives the result for all t.

Mathematical Solution:

1. **Compute $H|\psi(0)\rangle$:**

$$H|\psi(0)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} E_a \\ E_b \\ A \end{pmatrix}$$

2. **Compute the inner product $\langle\psi(0)|H|\psi(0)\rangle$:**

$$\langle\psi(0)|H|\psi(0)\rangle = \frac{1}{2}(E_a + E_b)$$

The expectation value of the energy is $\frac{E_a+E_b}{2}$, independent of A and t.

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    return (E_a + E_b) / 2.0
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.553348+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (['2025-02-08T21:40:21.790829+00:00']) Grade: A

Notes: The submitted solution correctly identifies that the expectation value of the energy is time-independent due to the time-independent Hamiltonian. The solution accurately computes the expectation value $\langle\psi(0)|H|\psi(0)\rangle$ and arrives at the correct result $\frac{1}{2}(E_a + E_b)$, which matches the standard solution. The reasoning is thorough, and the solution process is consistent with quantum mechanical principles. The inclusion of a Python function to compute the result is a valid extension, although it is not necessary for the mathematical solution.

Attempt 4 (2025-02-08T06:23:27.650003+00:00)

Detailed Solution:

The expectation value of the energy at any time t is determined by evaluating $\langle\psi(0)|H|\psi(0)\rangle$. Since the Hamiltonian is time-independent, this expectation value remains constant over time.

Step-by-Step Explanation: 1. **Initial State:** The normalized initial state is $\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

2. **Hamiltonian Application:** Compute $H\psi(0)$:

$$H\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} E_a \cdot 1 + 0 \cdot 1 + A \cdot 0 \\ 0 \cdot 1 + E_b \cdot 1 + 0 \cdot 0 \\ A \cdot 1 + 0 \cdot 1 + E_a \cdot 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} E_a \\ E_b \\ A \end{pmatrix}.$$

3. **Inner Product:** Calculate $\langle\psi(0)|H|\psi(0)\rangle$:

$$\langle\psi(0)|H|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 & 0\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}E_a \\ E_b \\ A\end{pmatrix} = \frac{1}{2}(E_a + E_b + 0) = \frac{E_a + E_b}{2}.$$

Final Answer: The expectation value of the energy is time-independent and equals $\frac{E_a+E_b}{2}$.

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    return (E_a + E_b) / 2.0
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.553701+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (['2025-02-08T21:40:44.163223+00:00']) Grade: A
Notes: The submitted solution correctly identifies that the expectation value of the energy is time-independent due to the time-independent Hamiltonian. The solution accurately computes the expectation value $\langle\psi(0)|H|\psi(0)\rangle$ using the initial state $\psi(0) = \frac{1}{\sqrt{2}}(1,1,0)^T$ and the given Hamiltonian. The calculation steps are detailed and correctly lead to the final result $\frac{E_a+E_b}{2}$, which matches the standard solution. The reasoning about the role of the parameter A and its cancellation due to the initial state's zero component is also correctly addressed. The Python function provided correctly implements the derived result.

Attempt 5 (2025-02-08T06:24:02.633732+00:00)

Detailed Solution:
The expectation value of the Hamiltonian H is time-independent and can be calculated directly using the initial state $\psi(0)$. The result is:

$$\langle H\rangle = \frac{E_a + E_b}{2}$$

This value remains constant for all t, independent of A.

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    return (E_a + E_b) / 2.0
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.554054+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (['2025-02-08T21:40:50.629074+00:00']) Grade: A
Notes: The submitted solution correctly identifies that the expectation value of the Hamiltonian is time-independent and calculates it using the initial state $\psi(0)$. The solution accurately computes $\langle H\rangle = \frac{E_a+E_b}{2}$, matching the standard solution. The reasoning is thorough, and the Python function provided correctly implements this result. The solution process includes all necessary steps and demonstrates a correct understanding of the problem.

2.2.4 Model: chatgpt-4o-latest

Attempt 1 (2025-02-02T04:22:11.538499Z)

Detailed Solution: Step 1: Understand the Problem We are tasked with finding the expectation value of the energy for a three-level quantum system at time t. The Hamiltonian H and the initial state $\psi(t=0)$ are given. The expectation value of the energy is calculated as:

$$\langle E \rangle(t) = \langle \psi(t) | H | \psi(t) \rangle,$$

where $\psi(t)$ is the time-evolved state of the system.
The time evolution of the state is governed by the Schrdinger equation:

$$\psi(t) = e^{-iHt/\hbar}\psi(0).$$

Step 2: Devise a Plan 1. **Diagonalize the Hamiltonian** H: Find the eigenvalues and eigenvectors of H. 2. **Express the time-evolved state** $\psi(t)$: Use the eigenvalues and eigenvectors to compute $e^{-iHt/\hbar}\psi(0)$. 3. **Compute the expectation value** $\langle E \rangle(t)$: Use the formula $\langle E \rangle(t) = \langle \psi(t) | H | \psi(t) \rangle$.
Step 3: Carry Out the Plan
3.1 Diagonalize the Hamiltonian The Hamiltonian is:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}.$$

The eigenvalues of H can be found by solving the characteristic equation:

$$\det(H - \lambda I) = 0.$$

This gives:

$$\det \begin{pmatrix} E_a - \lambda & 0 & A \\ 0 & E_b - \lambda & 0 \\ A & 0 & E_a - \lambda \end{pmatrix} = 0.$$

Expanding the determinant:

$$(E_a - \lambda) \det \begin{pmatrix} E_b - \lambda & 0 \\ 0 & E_a - \lambda \end{pmatrix} - A^2 = 0.$$

The determinant simplifies to:

$$(E_a - \lambda)((E_b - \lambda)(E_a - \lambda)) - A^2 = 0.$$

Expanding further:

$$(E_a - \lambda)^2 (E_b - \lambda) - A^2 = 0.$$

Let us denote the eigenvalues as $\lambda_1, \lambda_2, \lambda_3$. Solving this cubic equation gives:

$$\lambda_1 = E_b, \quad \lambda_2 = E_a + A, \quad \lambda_3 = E_a - A.$$

The eigenvectors corresponding to these eigenvalues can be computed, but for brevity, we will use the fact that H is symmetric and can be diagonalized by an orthogonal matrix.
3.2 Time Evolution of the State The initial state is:

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

The time-evolved state is:

$$\psi(t) = e^{-iHt/\hbar}\psi(0).$$

Using the eigenvalues and eigenvectors of H, we can write:

$$\psi(t) = \sum_{j=1}^{3} c_j e^{-i\lambda_j t/\hbar} v_j,$$

where v_j are the eigenvectors of H, and $c_j = v_j^\dagger \psi(0)$ are the expansion coefficients of $\psi(0)$ in the eigenbasis.

3.3 Expectation Value of Energy The expectation value of the energy is:

$$\langle E \rangle(t) = \langle \psi(t)|H|\psi(t) \rangle.$$

Substituting $\psi(t)$, we compute this explicitly.
Final Symbolic Expression After performing the detailed calculations (omitted here for brevity), the expectation value simplifies to:

$$\langle E \rangle(t) = \frac{1}{2}(E_a + E_b).$$

Step 4: Python Code Implementation We now implement the solution in Python.

```
        import numpy as np

def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    """
    Compute the expectation value of the energy for the given parameters.

    Parameters:
    A (float): Coupling constant in the Hamiltonian.
    E_a (float): Energy level of the first and third states.
    E_b (float): Energy level of the second state.
    t (float): Time (not relevant for this specific problem).

    Returns:
    float: The expectation value of the energy.
    """
    # The expectation value is independent of time and simplifies to:
    return 0.5 * (E_a + E_b)
```

Final Answer: The expectation value of the energy is:

$$\langle E \rangle(t) = \frac{1}{2}(E_a + E_b).$$

Verification Results:

Auto verification result (['2025-02-18T05:50:02.544149+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:59:45.905749Z) Grade: A
Notes: The submitted solution correctly follows the steps to find the expectation value of the energy for the given quantum system. It includes the diagonalization of the Hamiltonian, the time evolution of the state, and the calculation of the expectation value. The final answer, $\langle E \rangle(t) = \frac{1}{2}(E_a + E_b)$, matches the standard solution. The approach is thorough and demonstrates a clear understanding of the problem, even though some intermediate calculations are omitted for brevity. The inclusion of a Python implementation is a valid extension, although not necessary for the solution.

Attempt 2 (2025-02-02T04:22:37.065052Z)

Detailed Solution: Step 1: Understand the Problem
We are tasked with finding the expectation value of the energy for a three-level quantum system described by the Hamiltonian:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix},$$

where A, E_a, and E_b are real constants. The initial state of the system is given as:

$$\psi(t=0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}.$$

We are required to compute the expectation value of the energy at time t, which is given by:

$$\langle E\rangle(t) = \langle\psi(t)|H|\psi(t)\rangle.$$

To do this, we need to: 1. Solve for the time-evolved state $\psi(t)$ using the Schrdinger equation. 2. Compute the expectation value of the Hamiltonian in the state $\psi(t)$.

—

Step 2: Solve for the Time-Evolved State $\psi(t)$

The time evolution of the quantum state is given by:

$$\psi(t) = e^{-iHt}\psi(0),$$

where e^{-iHt} is the time evolution operator. To compute this, we first diagonalize the Hamiltonian H.

Step 2.1: Diagonalize the Hamiltonian

The eigenvalues of H are found by solving the characteristic equation:

$$\det(H - \lambda I) = 0.$$

Substituting H, we have:

$$\det\begin{pmatrix}E_a-\lambda & 0 & A\\ 0 & E_b-\lambda & 0\\ A & 0 & E_a-\lambda\end{pmatrix} = 0.$$

Expanding the determinant, we get:

$$(E_a-\lambda)\det\begin{pmatrix}E_b-\lambda & 0\\ 0 & E_a-\lambda\end{pmatrix} - A\det\begin{pmatrix}0 & A\\ A & E_b-\lambda\end{pmatrix} = 0.$$

The determinant simplifies to:

$$(E_a-\lambda)((E_b-\lambda)(E_a-\lambda)) - A^2(E_b-\lambda) = 0.$$

Factoring out $(E_b-\lambda)$, we get:

$$(E_b-\lambda)((E_a-\lambda)^2 - A^2) = 0.$$

Thus, the eigenvalues are:

$$\lambda_1 = E_b, \quad \lambda_2 = E_a + A, \quad \lambda_3 = E_a - A.$$

Step 2.2: Construct the Time Evolution Operator

The Hamiltonian can be written in terms of its eigenvalues and eigenvectors as:

$$H = U\Lambda U^\dagger,$$

where U is the matrix of eigenvectors, and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ is the diagonal matrix of eigenvalues. The time evolution operator is then:

$$e^{-iHt} = Ue^{-i\Lambda t}U^\dagger,$$

where $e^{-i\Lambda t} = \mathrm{diag}(e^{-i\lambda_1 t}, e^{-i\lambda_2 t}, e^{-i\lambda_3 t})$.

Step 2.3: Apply the Time Evolution to $\psi(0)$

The time-evolved state is:

$$\psi(t) = e^{-iHt}\psi(0).$$

—

Step 3: Compute the Expectation Value of Energy

The expectation value of the energy is:

$$\langle E\rangle(t) = \langle\psi(t)|H|\psi(t)\rangle.$$

Substituting $\psi(t) = e^{-iHt}\psi(0)$, we have:

$$\langle E\rangle(t) = \psi(0)^\dagger e^{iHt} H e^{-iHt} \psi(0).$$

Using the fact that $e^{iHt} H e^{-iHt} = H$, this simplifies to:

$$\langle E\rangle(t) = \psi(0)^\dagger H \psi(0).$$

Thus, the expectation value depends only on the initial state $\psi(0)$ and the Hamiltonian H.

—

Step 4: Compute $\langle E\rangle(t)$ Explicitly

Substitute $\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$ into the expression for $\langle E\rangle(t)$:

$$\langle E\rangle(t) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 & 0\end{pmatrix}\begin{pmatrix}E_a & 0 & A\\ 0 & E_b & 0\\ A & 0 & E_a\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}.$$

Perform the matrix multiplication:

$$H\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}E_a + E_b\\ E_b\\ A\end{pmatrix}.$$

Taking the inner product:

$$\langle E\rangle(t) = \frac{1}{2}\begin{pmatrix}1 & 1 & 0\end{pmatrix}\begin{pmatrix}E_a + E_b\\ E_b\\ A\end{pmatrix}.$$

Simplify the expression:

$$\langle E\rangle(t) = \frac{1}{2}\left[(E_a + E_b) + E_b\right] = \frac{1}{2}(E_a + 2E_b).$$

—

Final Answer:

$$\langle E\rangle(t) = \frac{1}{2}(E_a + 2E_b).$$

—

Step 5: Python Code Implementation

Below is the Python implementation of the solution:

```
def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    """
    Compute the expectation value of the energy for the given parameters.

    Parameters:
    A (float): Coupling constant (not used in the final result).
    E_a (float): Energy level of the first state.
    E_b (float): Energy level of the second state.
    t (float): Time (not used in the final result).

    Returns:
    float: The expectation value of the energy.
    """
    return 0.5 * (E_a + 2 * E_b)
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.544534+00:00']) Status: **Incorrect**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:59:52.297833Z) Grade: C
Notes: The submitted solution shows a basic understanding of the problem but contains significant errors. The main issue is in the calculation of the expectation value of energy. The final expression $\langle E\rangle(t) = \frac{1}{2}(E_a + 2E_b)$ is incorrect. The correct expectation value should be $\langle E\rangle = \frac{1}{2}(E_a + E_b)$ as derived in the standard solution. The error arises from the incorrect application of the Hamiltonian to the initial state and the subsequent inner product calculation. The approach to diagonalize the Hamiltonian and consider time evolution is valid, but the execution has flaws leading to the wrong final result.

Attempt 3 (2025-02-02T04:23:04.786026Z)

Detailed Solution: Step 1: Understand the Problem
We are tasked with finding the expectation value of the energy for a three-level quantum system at time t. The Hamiltonian of the system is given as:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix},$$

where A is a real constant. The initial state of the system is:

$$\psi(t=0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

The goal is to compute the expectation value of the energy at time t, which is given by:

$$\langle E\rangle(t) = \langle \psi(t)|H|\psi(t)\rangle.$$

Step 2: Devise a Plan
1. **Solve the time evolution of the state**: The time-evolved state is given by:

$$\psi(t) = e^{-iHt}\psi(0),$$

where e^{-iHt} is the time evolution operator.
2. **Compute the expectation value**: The expectation value of the energy is:

$$\langle E\rangle(t) = \psi(t)^\dagger H \psi(t).$$

3. **Diagonalize the Hamiltonian**: To compute e^{-iHt}, we diagonalize H as $H = UDU^\dagger$, where D is the diagonal matrix of eigenvalues and U is the matrix of eigenvectors. Then:

$$e^{-iHt} = Ue^{-iDt}U^\dagger.$$

4. **Substitute and simplify**: Substitute $\psi(t)$ into the expression for $\langle E\rangle(t)$ and simplify.
—
Step 3: Carry Out the Plan
Step 3.1: Diagonalize the Hamiltonian
The Hamiltonian is:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}.$$

To find the eigenvalues, we solve $\det(H - \lambda I) = 0$, where λ is the eigenvalue. The characteristic equation is:

$$\det \begin{pmatrix} E_a - \lambda & 0 & A \\ 0 & E_b - \lambda & 0 \\ A & 0 & E_a - \lambda \end{pmatrix} = 0.$$

Expanding the determinant:

$$(E_a - \lambda)^2(E_b - \lambda) - A^2(E_b - \lambda) = 0.$$

Factorizing:

$$(E_b - \lambda)\left[(E_a - \lambda)^2 - A^2\right] = 0.$$

This gives the eigenvalues:

$$\lambda_1 = E_b, \quad \lambda_2 = E_a + A, \quad \lambda_3 = E_a - A.$$

Thus, the diagonal matrix D is:

$$D = \begin{pmatrix} E_b & 0 & 0 \\ 0 & E_a + A & 0 \\ 0 & 0 & E_a - A \end{pmatrix}.$$

The eigenvectors corresponding to these eigenvalues can be computed, but for brevity, we denote the matrix of eigenvectors as U, such that:

$$H = UDU^\dagger.$$

Step 3.2: Time Evolution of the State
The time-evolved state is:

$$\psi(t) = e^{-iHt}\psi(0) = Ue^{-iDt}U^\dagger\psi(0).$$

Step 3.3: Compute the Expectation Value
The expectation value of the energy is:

$$\langle E\rangle(t) = \psi(t)^\dagger H\psi(t).$$

Substituting $\psi(t) = Ue^{-iDt}U^\dagger\psi(0)$, we get:

$$\langle E\rangle(t) = \psi(0)^\dagger Ue^{iDt}U^\dagger HUe^{-iDt}U^\dagger\psi(0).$$

Using $H = UDU^\dagger$, this simplifies to:

$$\langle E\rangle(t) = \psi(0)^\dagger UDU^\dagger\psi(0).$$

Let $\phi = U^\dagger\psi(0)$. Then:

$$\langle E\rangle(t) = \phi^\dagger D\phi.$$

Explicitly, D is diagonal, so:

$$\langle E\rangle(t) = |c_1|^2 E_b + |c_2|^2(E_a + A) + |c_3|^2(E_a - A),$$

where c_1, c_2, c_3 are the components of ϕ.
Step 3.4: Initial State Transformation
The initial state is:

$$\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Transforming to the eigenbasis:

$$\phi = U^\dagger\psi(0).$$

The coefficients c_1, c_2, c_3 can be computed explicitly, but the final result depends on the specific form of U.
—
Final Answer (Symbolic Expression)
The expectation value of the energy at time t is:

$$\langle E\rangle(t) = |c_1|^2 E_b + |c_2|^2(E_a + A) + |c_3|^2(E_a - A),$$

where c_1, c_2, c_3 are the components of $\phi = U^\dagger\psi(0)$.
—
Step 4: Python Code Implementation
Below is the Python implementation to compute the expectation value.

```python
        import numpy as np

def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    # Define the Hamiltonian
    H = np.array([
        [E_a, 0, A],
        [0, E_b, 0],
        [A, 0, E_a]
    ])

    # Initial state
    psi_0 = np.array([1/np.sqrt(2), 1/np.sqrt(2), 0])

    # Diagonalize the Hamiltonian
    eigenvalues, U = np.linalg.eigh(H)
    D = np.diag(eigenvalues)

    # Time evolution operator
    exp_minus_iDt = np.diag(np.exp(-1j * eigenvalues * t))
    exp_iDt = np.diag(np.exp(1j * eigenvalues * t))

    # Time-evolved state
    psi_t = U @ exp_minus_iDt @ U.T.conj() @ psi_0

    # Compute the expectation value
    expectation = np.real(psi_t.conj().T @ H @ psi_t)

    return expectation
```

This function computes the expectation value of the energy at time t for the given parameters A, E_a, and E_b.
Verification Results:

Auto verification result (['2025-02-18T05:50:02.545464+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (2025-02-04T03:00:00.210584Z) Grade: C
Notes: The submitted solution demonstrates a basic understanding of the problem by attempting to find the expectation value of energy for a quantum system. However, it has significant issues. The solution does not correctly address the decomposition of the initial state into the eigenbasis of the Hamiltonian, which is crucial for finding the expectation value. The final expression for the expectation value is left in a symbolic form involving coefficients c_1, c_2, c_3 without computing them, which is necessary to match the standard solution. The standard solution directly computes the expectation value using the given decomposition of the state, resulting in $\langle E \rangle = \frac{1}{2}(E_a + E_b)$. The submitted solution's approach is overly complex and does not reach the correct final answer.

Attempt 4 (2025-02-02T04:23:27.801234Z)

Detailed Solution: Step 1: Understand the Problem We are tasked with finding the expectation value of the energy for a three-level quantum system at time t. The Hamiltonian of the system is given as:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix},$$

and the initial state of the system is:

$$\psi(t = 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

The expectation value of the energy at time t is given by:

$$\langle E\rangle(t) = \langle\psi(t)|H|\psi(t)\rangle,$$

where $\psi(t)$ is the time-evolved state of the system.
Step 2: Devise a Plan 1. **Diagonalize the Hamiltonian**: To compute the time evolution of the state, we need the eigenvalues and eigenvectors of H, as the time evolution operator $e^{-iHt/\hbar}$ is diagonal in the eigenbasis of H. 2. **Compute the time-evolved state**: Use the eigenvalues and eigenvectors to express $\psi(t)$ in terms of the initial state $\psi(0)$. 3. **Calculate the expectation value**: Substitute $\psi(t)$ into the expression for $\langle E\rangle(t)$ and simplify.

—

Step 3: Solve the Problem
3.1 Diagonalize the Hamiltonian The Hamiltonian H is:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}.$$

Let us find the eigenvalues λ by solving the characteristic equation $\det(H - \lambda I) = 0$:

$$\det\begin{pmatrix} E_a - \lambda & 0 & A \\ 0 & E_b - \lambda & 0 \\ A & 0 & E_a - \lambda \end{pmatrix} = 0.$$

Expanding the determinant:

$$\det(H - \lambda I) = (E_a - \lambda)\det\begin{pmatrix} E_b - \lambda & 0 \\ 0 & E_a - \lambda \end{pmatrix} - A^2.$$

The determinant of the 2×2 block is:

$$\det\begin{pmatrix} E_b - \lambda & 0 \\ 0 & E_a - \lambda \end{pmatrix} = (E_b - \lambda)(E_a - \lambda).$$

Thus:

$$\det(H - \lambda I) = (E_a - \lambda)\left[(E_b - \lambda)(E_a - \lambda)\right] - A^2.$$

Simplify:

$$\det(H - \lambda I) = (E_a - \lambda)^2(E_b - \lambda) - A^2.$$

This cubic equation gives the eigenvalues $\lambda_1, \lambda_2, \lambda_3$. For simplicity, we denote the eigenvalues symbolically as $\lambda_1, \lambda_2, \lambda_3$.
3.2 Time Evolution of the State The time evolution of the state is given by:

$$\psi(t) = e^{-iHt/\hbar}\psi(0).$$

In the eigenbasis of H, the time evolution operator is diagonal:

$$e^{-iHt/\hbar} = \sum_{j=1}^{3} e^{-i\lambda_j t/\hbar}|\phi_j\rangle\langle\phi_j|,$$

where $|\phi_j\rangle$ are the eigenvectors of H corresponding to eigenvalues λ_j.
Express the initial state $\psi(0)$ in terms of the eigenvectors:

$$\psi(0) = \sum_{j=1}^{3} c_j|\phi_j\rangle,$$

where $c_j = \langle\phi_j|\psi(0)\rangle$. Then the time-evolved state is:

$$\psi(t) = \sum_{j=1}^{3} c_j e^{-i\lambda_j t/\hbar}|\phi_j\rangle.$$

3.3 Expectation Value of Energy The expectation value of the energy is:

$$\langle E \rangle(t) = \langle \psi(t) | H | \psi(t) \rangle.$$

Substitute $\psi(t)$:

$$\langle E \rangle(t) = \sum_{j,k} c_j^* c_k e^{i(\lambda_j - \lambda_k)t/\hbar} \langle \phi_j | H | \phi_k \rangle.$$

Since $H|\phi_k\rangle = \lambda_k |\phi_k\rangle$, we have:

$$\langle \phi_j | H | \phi_k \rangle = \lambda_k \delta_{jk}.$$

Thus:

$$\langle E \rangle(t) = \sum_j |c_j|^2 \lambda_j.$$

The coefficients c_j are determined from the initial state $\psi(0)$:

$$c_j = \langle \phi_j | \psi(0) \rangle.$$

Finally:

$$\langle E \rangle(t) = \sum_j |\langle \phi_j | \psi(0) \rangle|^2 \lambda_j.$$

—

Step 4: Python Code Implementation
Below is the Python implementation of the derived formula.

```
        import numpy as np

def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    # Define the Hamiltonian
    H = np.array([
        [E_a, 0, A],
        [0, E_b, 0],
        [A, 0, E_a]
    ])

    # Diagonalize the Hamiltonian
    eigenvalues, eigenvectors = np.linalg.eigh(H)

    # Initial state
    psi_0 = np.array([1/np.sqrt(2), 1/np.sqrt(2), 0])

    # Compute the coefficients c_j = <phi_j | psi(0)>
    c = np.dot(eigenvectors.T.conj(), psi_0)

    # Compute the expectation value
    expectation = np.sum(np.abs(c)**2 * eigenvalues)

    return float(expectation)
```

—

Final Answer: The expectation value of the energy at time t is computed using the above Python function.
Verification Results:

Auto verification result (['2025-02-18T05:50:02.546123+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (2025-02-04T03:00:06.363980Z) Grade: C
Notes: The submitted solution demonstrates a basic understanding of the problem by attempting to diagonalize the Hamiltonian and compute the expectation value of energy. However, it contains significant issues. The solution does not explicitly find the eigenstates and eigenvalues, which are crucial for the correct decomposition of the initial state and calculation of the expectation value. The standard solution provides specific

eigenstates and energies, which are not derived in the submitted solution. Additionally, the final answer is not explicitly stated in terms of the given parameters E_a, E_b, and A, and the Python code implementation is not directly linked to the analytical solution. The approach has merit, but the execution is incomplete and lacks crucial steps.

Attempt 5 (2025-02-02T04:23:56.549374Z)

Detailed Solution: Step 1: Understand the Problem We are tasked with finding the expectation value of the energy at time t for a three-level quantum system. The Hamiltonian H and the initial state $\psi(t=0)$ are provided. The expectation value of the energy is given by:

$$\langle E(t)\rangle = \langle \psi(t)|H|\psi(t)\rangle$$

where $\psi(t)$ is the time-evolved state of the system:

$$\psi(t) = e^{-iHt/\hbar}\psi(0).$$

Step 2: Devise a Plan 1. **Diagonalize the Hamiltonian H:** Find its eigenvalues and eigenvectors. 2. **Express the time-evolved state $\psi(t)$:** Use the eigenvalues and eigenvectors to compute $e^{-iHt/\hbar}\psi(0)$. 3. **Compute the expectation value $\langle E(t)\rangle$:** Use the formula $\langle E(t)\rangle = \langle \psi(t)|H|\psi(t)\rangle$.
Step 3: Carry Out the Plan
Step 3.1: Diagonalize the Hamiltonian The Hamiltonian is:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}.$$

We solve the eigenvalue equation $Hv = \lambda v$, where λ are the eigenvalues.
The characteristic equation is:

$$\det(H - \lambda I) = 0.$$

Substituting $H - \lambda I$:

$$\det\begin{pmatrix} E_a-\lambda & 0 & A \\ 0 & E_b-\lambda & 0 \\ A & 0 & E_a-\lambda \end{pmatrix} = 0.$$

Expanding the determinant:

$$\det(H - \lambda I) = (E_a-\lambda)^2(E_b-\lambda) - A^2(E_b-\lambda).$$

Factorizing:

$$\det(H - \lambda I) = (E_b-\lambda)\left[(E_a-\lambda)^2 - A^2\right].$$

Expanding $(E_a-\lambda)^2 - A^2$:

$$(E_a-\lambda)^2 - A^2 = (E_a-\lambda-A)(E_a-\lambda+A).$$

Thus, the eigenvalues are:

$$\lambda_1 = E_b, \quad \lambda_2 = E_a + A, \quad \lambda_3 = E_a - A.$$

Step 3.2: Eigenvectors of H For $\lambda_1 = E_b$, the eigenvector satisfies:

$$\begin{pmatrix} E_a-E_b & 0 & A \\ 0 & 0 & 0 \\ A & 0 & E_a-E_b \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0.$$

This gives $v_2 = 1$, $v_1 = v_3 = 0$. Thus, the eigenvector is:

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

For $\lambda_2 = E_a + A$, the eigenvector satisfies:

$$\begin{pmatrix} -A & 0 & A \\ 0 & E_b - (E_a + A) & 0 \\ A & 0 & -A \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0.$$

Solving, we find:

$$v_2 = 0, \quad v_1 = v_3.$$

Thus, the eigenvector is:

$$v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda_3 = E_a - A$, the eigenvector satisfies:

$$\begin{pmatrix} A & 0 & A \\ 0 & E_b - (E_a - A) & 0 \\ A & 0 & A \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0.$$

Solving, we find:

$$v_2 = 0, \quad v_1 = -v_3.$$

Thus, the eigenvector is:

$$v_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Step 3.3: Time-Evolved State The initial state is:

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Express $\psi(0)$ in the eigenbasis:

$$\psi(0) = c_1 v_1 + c_2 v_2 + c_3 v_3.$$

Projecting onto the eigenvectors:

$$c_1 = \langle v_1 | \psi(0) \rangle = \frac{1}{\sqrt{2}}, \quad c_2 = \langle v_2 | \psi(0) \rangle = \frac{1}{2}, \quad c_3 = \langle v_3 | \psi(0) \rangle = \frac{1}{2}.$$

Thus:

$$\psi(0) = \frac{1}{\sqrt{2}} v_1 + \frac{1}{2} v_2 + \frac{1}{2} v_3.$$

The time-evolved state is:

$$\psi(t) = \frac{1}{\sqrt{2}} e^{-iE_b t/\hbar} v_1 + \frac{1}{2} e^{-i(E_a + A)t/\hbar} v_2 + \frac{1}{2} e^{-i(E_a - A)t/\hbar} v_3.$$

Step 3.4: Expectation Value of Energy The expectation value is:

$$\langle E(t) \rangle = \langle \psi(t) | H | \psi(t) \rangle.$$

Substituting $\psi(t)$ and simplifying, we find:

$$\langle E(t) \rangle = \frac{E_b}{2} + \frac{E_a + A}{4} + \frac{E_a - A}{4} = \frac{E_a}{2} + \frac{E_b}{2}.$$

Final Answer:

$$\langle E(t) \rangle = \frac{E_a + E_b}{2}.$$

—

Python Code Implementation

```
        import numpy as np

def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    # The expectation value is independent of time
    return (E_a + E_b) / 2
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.546489+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (2025-02-04T03:00:12.254283Z) Grade: A

Notes: The submitted solution correctly follows the steps to find the eigenvalues and eigenvectors of the Hamiltonian, expresses the initial state in terms of these eigenvectors, and calculates the time-evolved state. The expectation value of the energy is computed accurately, matching the standard solution: $\langle E(t) \rangle = \frac{E_a+E_b}{2}$. The solution process is thorough and demonstrates a clear understanding of the quantum mechanics involved, including the use of the eigenbasis for time evolution and expectation value calculation. The final answer is equivalent to the standard solution.

2.2.5 Model: deepseek-ai/DeepSeek-V3

Attempt 1 (2025-02-06T20:04:22.104749+00:00)

Detailed Solution: Mathematical Solution

Step 1: Understand the Problem We are given the Hamiltonian H of a three-level system and the initial state $\psi(t=0)$. We need to find the expectation value of the energy at time t.

Step 2: Diagonalize the Hamiltonian The Hamiltonian is given by:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

To find the expectation value of the energy, we need to diagonalize H. The eigenvalues λ of H are found by solving the characteristic equation $\det(H - \lambda I) = 0$.

The characteristic equation is:

$$\det \begin{pmatrix} E_a - \lambda & 0 & A \\ 0 & E_b - \lambda & 0 \\ A & 0 & E_a - \lambda \end{pmatrix} = 0$$

Expanding the determinant, we get:

$$(E_a - \lambda)\left[(E_a - \lambda)(E_b - \lambda) - A^2\right] = 0$$

This gives us the eigenvalues:

$$\lambda_1 = E_b, \quad \lambda_2 = E_a + A, \quad \lambda_3 = E_a - A$$

The corresponding eigenvectors are:

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Step 3: Express the Initial State in the Eigenbasis The initial state is:

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We express $\psi(0)$ in terms of the eigenvectors:

$$\psi(0) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

Solving for the coefficients c_1, c_2, c_3:

$$c_1 = \langle \mathbf{v}_1 | \psi(0) \rangle = \frac{1}{\sqrt{2}}, \quad c_2 = \langle \mathbf{v}_2 | \psi(0) \rangle = \frac{1}{2}, \quad c_3 = \langle \mathbf{v}_3 | \psi(0) \rangle = \frac{1}{2}$$

Step 4: Time Evolution of the State The time-evolved state is:

$$\psi(t) = c_1 e^{-i\lambda_1 t}\mathbf{v}_1 + c_2 e^{-i\lambda_2 t}\mathbf{v}_2 + c_3 e^{-i\lambda_3 t}\mathbf{v}_3$$

Substituting the coefficients and eigenvalues:

$$\psi(t) = \frac{1}{\sqrt{2}} e^{-iE_b t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{2} e^{-i(E_a + A)t} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{2} e^{-i(E_a - A)t} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Step 5: Expectation Value of the Energy The expectation value of the energy is:

$$\langle E \rangle = \langle \psi(t) | H | \psi(t) \rangle$$

Since H is diagonal in the eigenbasis, we can write:

$$\langle E \rangle = |c_1|^2 \lambda_1 + |c_2|^2 \lambda_2 + |c_3|^2 \lambda_3$$

Substituting the values:

$$\langle E \rangle = \left(\frac{1}{\sqrt{2}}\right)^2 E_b + \left(\frac{1}{2}\right)^2 (E_a + A) + \left(\frac{1}{2}\right)^2 (E_a - A)$$

Simplifying:

$$\langle E \rangle = \frac{1}{2}E_b + \frac{1}{4}(E_a + A) + \frac{1}{4}(E_a - A) = \frac{1}{2}E_b + \frac{1}{2}E_a$$

Final Answer:

$$\langle E \rangle = \frac{1}{2}(E_a + E_b)$$

Python Implementation

```
def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    # The expectation value of the energy is time-independent
    return 0.5 * (E_a + E_b)
```

This Python function calculates the expectation value of the energy at any time t, which is constant and given by $\frac{1}{2}(E_a + E_b)$.
Verification Results:

Auto verification result (['2025-02-18T05:50:02.550792+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (['2025-02-07T22:43:56.182965+00:00']) Grade: A
Notes: The submitted solution correctly follows the steps to diagonalize the Hamiltonian, find the eigenvalues and eigenvectors, express the initial state in the eigenbasis, and compute the expectation value of the energy. The solution matches the standard solution in terms of the final result, $\langle E \rangle = \frac{1}{2}(E_a + E_b)$, and the reasoning is sound. The approach is thorough and includes all necessary steps, demonstrating a clear understanding of the problem.

Attempt 2 (2025-02-06T20:07:19.572559+00:00)

Detailed Solution: Step 1: Understanding the Problem
We are given a three-level system with the Hamiltonian:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

where A is real. The initial state of the system at $t = 0$ is:

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We need to find the expectation value of the energy at time t.
Step 2: Devise a Plan
To find the expectation value of the energy at time t, we need to: 1. Find the eigenvalues and eigenvectors of the Hamiltonian H. 2. Express the initial state $\psi(0)$ in terms of the eigenstates of H. 3. Evolve the state $\psi(t)$ in time using the eigenvalues. 4. Calculate the expectation value of the energy $\langle \psi(t) | H | \psi(t) \rangle$.
Step 3: Carry Out the Plan
Step 3.1: Find the Eigenvalues and Eigenvectors of H
The Hamiltonian H is:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

To find the eigenvalues, we solve the characteristic equation $\det(H - \lambda I) = 0$:

$$\det\begin{pmatrix} E_a - \lambda & 0 & A \\ 0 & E_b - \lambda & 0 \\ A & 0 & E_a - \lambda \end{pmatrix} = 0$$

Expanding the determinant:

$$(E_a - \lambda)\det\begin{pmatrix} E_b - \lambda & 0 \\ 0 & E_a - \lambda \end{pmatrix} - A\det\begin{pmatrix} 0 & E_b - \lambda \\ A & 0 \end{pmatrix} = 0$$

$$(E_a - \lambda)\left[(E_b - \lambda)(E_a - \lambda)\right] - A\left[0 - A(E_b - \lambda)\right] = 0$$

$$(E_a - \lambda)^2(E_b - \lambda) + A^2(E_b - \lambda) = 0$$

$$(E_b - \lambda)\left[(E_a - \lambda)^2 + A^2\right] = 0$$

The eigenvalues are:

$$\lambda_1 = E_b, \quad \lambda_2 = E_a + A, \quad \lambda_3 = E_a - A$$

Next, we find the corresponding eigenvectors.
For $\lambda_1 = E_b$:

$$\begin{pmatrix} E_a - E_b & 0 & A \\ 0 & 0 & 0 \\ A & 0 & E_a - E_b \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

This gives x_2 as a free variable, and $x_1 = x_3 = 0$. So, the eigenvector is:

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_2 = E_a + A$:

$$\begin{pmatrix} -A & 0 & A \\ 0 & -A & 0 \\ A & 0 & -A \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

This gives $x_2 = 0$ and $x_1 = x_3$. So, the eigenvector is:

$$v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda_3 = E_a - A$:

$$\begin{pmatrix} A & 0 & A \\ 0 & A & 0 \\ A & 0 & A \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

This gives $x_2 = 0$ and $x_1 = -x_3$. So, the eigenvector is:

$$v_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Step 3.2: Express $\psi(0)$ in Terms of Eigenstates
The initial state is:

$$\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We express $\psi(0)$ as a linear combination of the eigenstates:

$$\psi(0) = c_1 v_1 + c_2 v_2 + c_3 v_3$$

Calculating the coefficients:

$$c_1 = \langle v_1 | \psi(0) \rangle = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$c_2 = \langle v_2|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} = \frac{1}{2}$$

$$c_3 = \langle v_3|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} = \frac{1}{2}$$

So,

$$\psi(0) = \frac{1}{\sqrt{2}}v_1 + \frac{1}{2}v_2 + \frac{1}{2}v_3$$

Step 3.3: Evolve the State $\psi(t)$ in Time
The time evolution of the state is:

$$\psi(t) = \frac{1}{\sqrt{2}}e^{-iE_bt}v_1 + \frac{1}{2}e^{-i(E_a+A)t}v_2 + \frac{1}{2}e^{-i(E_a-A)t}v_3$$

Step 3.4: Calculate the Expectation Value of the Energy
The expectation value of the energy is:

$$\langle\psi(t)|H|\psi(t)\rangle$$

Since H is Hermitian and v_i are eigenstates of H, we have:

$$\langle\psi(t)|H|\psi(t)\rangle = \left|\frac{1}{\sqrt{2}}\right|^2 E_b + \left|\frac{1}{2}\right|^2 (E_a + A) + \left|\frac{1}{2}\right|^2 (E_a - A)$$

$$= \frac{1}{2}E_b + \frac{1}{4}(E_a + A) + \frac{1}{4}(E_a - A)$$

$$= \frac{1}{2}E_b + \frac{1}{2}E_a$$

Thus, the expectation value of the energy at time t is:

$$\langle E\rangle = \frac{1}{2}E_a + \frac{1}{2}E_b$$

Final Answer:

$$\langle E\rangle = \frac{1}{2}E_a + \frac{1}{2}E_b$$

Python Code Implementation

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    return 0.5 * E_a + 0.5 * E_b
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.551132+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (['2025-02-07T22:44:01.843444+00:00']) Grade: A
Notes: The submitted solution correctly follows the steps to find the eigenvalues and eigenvectors of the Hamiltonian, expresses the initial state in terms of these eigenstates, evolves the state in time, and calculates the expectation value of the energy. The final answer, $\langle E\rangle = \frac{1}{2}E_a + \frac{1}{2}E_b$, matches the standard solution. The solution process is detailed and demonstrates a correct understanding of the problem.

Attempt 3 (2025-02-06T20:09:28.632380+00:00)

Detailed Solution: Step 1: Understand the Problem
We are given a three-level system with the Hamiltonian:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

The initial state of the system at $t = 0$ is:

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We need to find the expectation value of the energy at time t.
Step 2: Devise a Plan
To find the expectation value of the energy at time t, we need to:
1. **Diagonalize the Hamiltonian** H to find its eigenvalues and eigenvectors. 2. **Express the initial state** $\psi(0)$ in terms of the eigenstates of H. 3. **Evolve the state** $\psi(t)$ using the time evolution operator. 4. **Calculate the expectation value** of the Hamiltonian H in the state $\psi(t)$.
Step 3: Carry Out the Plan
Step 3.1: Diagonalize the Hamiltonian
The Hamiltonian H is:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

To find the eigenvalues, we solve the characteristic equation $\det(H - \lambda I) = 0$:

$$\det \begin{pmatrix} E_a - \lambda & 0 & A \\ 0 & E_b - \lambda & 0 \\ A & 0 & E_a - \lambda \end{pmatrix} = 0$$

Expanding the determinant:

$$(E_a - \lambda) \cdot \det \begin{pmatrix} E_b - \lambda & 0 \\ 0 & E_a - \lambda \end{pmatrix} - A \cdot \det \begin{pmatrix} 0 & 0 \\ A & E_a - \lambda \end{pmatrix} = 0$$

Simplifying:

$$(E_a - \lambda) \cdot (E_b - \lambda)(E_a - \lambda) - A \cdot 0 = 0$$

$$(E_a - \lambda)^2 (E_b - \lambda) = 0$$

Thus, the eigenvalues are:

$$\lambda_1 = E_a, \quad \lambda_2 = E_a, \quad \lambda_3 = E_b$$

Next, we find the eigenvectors corresponding to these eigenvalues.
For $\lambda = E_b$:

$$\begin{pmatrix} E_a - E_b & 0 & A \\ 0 & 0 & 0 \\ A & 0 & E_a - E_b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

This gives:

$$(E_a - E_b)x + Az = 0 Ax + (E_a - E_b)z = 0$$

Assuming $E_a \neq E_b$, the solution is $x = z = 0$, and y is free. Thus, the eigenvector is:

$$v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda = E_a$:

$$\begin{pmatrix} 0 & 0 & A \\ 0 & E_b - E_a & 0 \\ A & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

This gives:

$$Az = 0(E_b - E_a)y = 0Ax = 0$$

Assuming $A \neq 0$, we have $x = z = 0$, and y is free. Thus, the eigenvectors are:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Step 3.2: Express the Initial State in Terms of Eigenstates
The initial state is:

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We express $\psi(0)$ in terms of the eigenstates v_1, v_2, v_3:

$$\psi(0) = \frac{1}{\sqrt{2}} v_1 + \frac{1}{\sqrt{2}} v_3$$

Step 3.3: Evolve the State $\psi(t)$
The time evolution of the state is given by:

$$\psi(t) = \frac{1}{\sqrt{2}} e^{-iE_a t/\hbar} v_1 + \frac{1}{\sqrt{2}} e^{-iE_b t/\hbar} v_3$$

Step 3.4: Calculate the Expectation Value of the Hamiltonian
The expectation value of the Hamiltonian H in the state $\psi(t)$ is:

$$\langle H \rangle = \langle \psi(t) | H | \psi(t) \rangle$$

Substituting $\psi(t)$:

$$\langle H \rangle = \left(\frac{1}{\sqrt{2}} e^{iE_a t/\hbar} \langle v_1 | + \frac{1}{\sqrt{2}} e^{iE_b t/\hbar} \langle v_3 | \right) H \left(\frac{1}{\sqrt{2}} e^{-iE_a t/\hbar} | v_1 \rangle + \frac{1}{\sqrt{2}} e^{-iE_b t/\hbar} | v_3 \rangle \right)$$

Since $Hv_1 = E_a v_1$ and $Hv_3 = E_b v_3$:

$$\langle H \rangle = \frac{1}{2} E_a + \frac{1}{2} E_b$$

Thus, the expectation value of the energy at time t is:

$$\langle H \rangle = \frac{E_a + E_b}{2}$$

Final Answer:
The expectation value of the energy at time t is:

$$\boxed{\langle H \rangle = \frac{E_a + E_b}{2}}$$

Python Implementation:

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    return (E_a + E_b) / 2
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.551472+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (['2025-02-07T22:44:08.444910+00:00']) Grade: A
Notes: The submitted solution correctly follows the necessary steps to find the expectation value of the energy. It starts by diagonalizing the Hamiltonian to find the eigenvalues and eigenvectors, expresses the initial state in terms of these eigenstates, evolves the state over time, and calculates the expectation value of the Hamiltonian. The final result, $\langle H \rangle = \frac{E_a+E_b}{2}$, matches the standard solution. The approach is thorough and demonstrates a clear understanding of the problem, despite using a different initial state representation. The Python implementation is a nice addition, although not necessary for the solution grading.

Attempt 4 (2025-02-06T20:10:56.164507+00:00)

Detailed Solution: Step 1: Understanding the Problem
We are given a three-level system with the Hamiltonian:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

where A is real. The initial state of the system at $t = 0$ is:

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We need to find the expectation value of the energy at time t.
Step 2: Devising a Plan
To find the expectation value of the energy at time t, we need to: 1. Find the eigenvalues and eigenvectors of the Hamiltonian H. 2. Express the initial state $\psi(0)$ in terms of the eigenstates of H. 3. Evolve the state $\psi(t)$ in time using the eigenvalues. 4. Calculate the expectation value $\langle \psi(t)|H|\psi(t)\rangle$.
Step 3: Carrying Out the Plan
Step 3.1: Find the Eigenvalues and Eigenvectors of H
The Hamiltonian H is:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

The characteristic equation is:

$$\det(H - \lambda I) = 0$$

This gives:

$$\begin{vmatrix} E_a - \lambda & 0 & A \\ 0 & E_b - \lambda & 0 \\ A & 0 & E_a - \lambda \end{vmatrix} = 0$$

Expanding the determinant:

$$(E_a - \lambda) \begin{vmatrix} E_b - \lambda & 0 \\ 0 & E_a - \lambda \end{vmatrix} - A \begin{vmatrix} 0 & 0 \\ A & E_a - \lambda \end{vmatrix} = 0$$

Simplifying:

$$(E_a - \lambda)(E_b - \lambda)(E_a - \lambda) - A^2(E_a - \lambda) = 0$$
$$(E_a - \lambda)\left[(E_b - \lambda)(E_a - \lambda) - A^2\right] = 0$$

This gives the eigenvalues:

$$\lambda_1 = E_a, \quad \lambda_2 = \frac{E_a + E_b}{2} + \frac{\sqrt{(E_a - E_b)^2 + 4A^2}}{2}, \quad \lambda_3 = \frac{E_a + E_b}{2} - \frac{\sqrt{(E_a - E_b)^2 + 4A^2}}{2}$$

Step 3.2: Express $\psi(0)$ in Terms of Eigenstates
The initial state is:

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We need to express this state as a linear combination of the eigenstates of H.
Step 3.3: Evolve the State $\psi(t)$
The time evolution of the state is given by:

$$\psi(t) = e^{-iHt/\hbar}\psi(0)$$

Using the eigenvalues and eigenvectors, we can write:

$$\psi(t) = \sum_i c_i e^{-i\lambda_i t/\hbar}|\phi_i\rangle$$

where c_i are the coefficients of $\psi(0)$ in the eigenbasis.
Step 3.4: Calculate the Expectation Value
The expectation value of the energy is:

$$\langle\psi(t)|H|\psi(t)\rangle$$

This can be calculated using the eigenvalues and the coefficients c_i.
Step 4: Final Calculation
After performing the detailed calculations (which involve finding the eigenvectors, expressing $\psi(0)$ in the eigenbasis, and evolving the state), the expectation value of the energy at time t is:

$$\langle E\rangle = \frac{E_a + E_b}{2} + \frac{A^2}{\sqrt{(E_a - E_b)^2 + 4A^2}}\cos\left(\frac{\sqrt{(E_a - E_b)^2 + 4A^2}}{\hbar}t\right)$$

Final Answer:

$$\langle E\rangle = \frac{E_a + E_b}{2} + \frac{A^2}{\sqrt{(E_a - E_b)^2 + 4A^2}}\cos\left(\frac{\sqrt{(E_a - E_b)^2 + 4A^2}}{\hbar}t\right)$$

Python Implementation:

```
        import math

def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    hbar = 1.0  # Assuming hbar is 1 for simplicity
    delta = E_a - E_b
    sqrt_term = math.sqrt(delta**2 + 4 * A**2)
    cos_term = math.cos((sqrt_term / hbar) * t)
    expectation = (E_a + E_b) / 2 + (A**2 / sqrt_term) * cos_term
    return expectation
```

This Python function calculates the expectation value of the energy at time t based on the derived formula.
Verification Results:

Auto verification result (['2025-02-18T05:50:02.551898+00:00']) Status: **Incorrect**

Evaluations:

Evaluated by gpt-4o (['2025-02-07T22:44:16.312442+00:00']) Grade: C
Notes: The submitted solution shows a basic understanding of the problem by attempting to find the eigenvalues and eigenvectors of the Hamiltonian and expressing the initial state in terms of these eigenstates. However, it diverges significantly from the standard solution. The standard solution finds a time-independent expectation value of energy, $\langle E\rangle = \frac{1}{2}(E_a + E_b)$, while the submitted solution introduces a time-dependent term involving a cosine function, $\langle E\rangle = \frac{E_a+E_b}{2} + \frac{A^2}{\sqrt{(E_a-E_b)^2+4A^2}}\cos\left(\frac{\sqrt{(E_a-E_b)^2+4A^2}}{\hbar}t\right)$. This indicates a major flaw in the approach, as the expectation value should not depend on time for the given problem. The solution also lacks the correct decomposition of the initial state into the eigenstates as shown in the standard solution.

Attempt 5 (2025-02-06T20:11:39.803773+00:00)

Detailed Solution: Step 1: Understanding the Problem
We are given a three-level quantum system with the Hamiltonian:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

where A is a real constant. The initial state of the system at $t = 0$ is:

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We need to find the expectation value of the energy at time t.
Step 2: Devise a Plan
To find the expectation value of the energy at time t, we need to: 1. Solve the time-dependent Schrdinger equation to find the state $\psi(t)$. 2. Compute the expectation value $\langle \psi(t)|H|\psi(t) \rangle$.
Step 3: Carry Out the Plan
Step 3.1: Solve the Time-Dependent Schrdinger Equation
The time evolution of the state $\psi(t)$ is given by:

$$\psi(t) = e^{-iHt/\hbar} \psi(0)$$

where $\hbar$ is the reduced Planck constant. However, since we are interested in the expectation value of the Hamiltonian, we can use the fact that the Hamiltonian is time-independent, and thus the expectation value of the energy is conserved:

$$\langle \psi(t)|H|\psi(t) \rangle = \langle \psi(0)|H|\psi(0) \rangle$$

This simplifies our problem to calculating the expectation value of the Hamiltonian at $t = 0$.
Step 3.2: Compute the Expectation Value at $t = 0$
The expectation value of the Hamiltonian at $t = 0$ is:

$$\langle \psi(0)|H|\psi(0) \rangle = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \right) H \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right)$$

First, compute $H\psi(0)$:

$$H\psi(0) = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} E_a \\ E_b \\ A \end{pmatrix}$$

Now, compute the inner product:

$$\langle \psi(0)|H|\psi(0) \rangle = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \right) \left(\frac{1}{\sqrt{2}} \begin{pmatrix} E_a \\ E_b \\ A \end{pmatrix} \right) = \frac{1}{2}(E_a + E_b)$$

Final Answer:
The expectation value of the energy at time t is:

$$\langle \psi(t)|H|\psi(t) \rangle = \frac{E_a + E_b}{2}$$

Python Implementation:

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    return (E_a + E_b) / 2
```

This function calculates the expectation value of the energy at any time t, which is constant and equal to the average of E_a and E_b.
Verification Results:

Auto verification result (['2025-02-18T05:50:02.552247+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (['2025-02-07T22:44:24.681523+00:00']) Grade: A

Notes: The submitted solution correctly identifies that the expectation value of the energy is conserved due to the time-independence of the Hamiltonian. It calculates the expectation value at $t = 0$ using the initial state $\psi(0) = \frac{1}{\sqrt{2}}(1, 1, 0)^T$ and the given Hamiltonian. The computation of $H\psi(0)$ and the subsequent inner product are performed correctly, leading to the final result $\langle\psi(t)|H|\psi(t)\rangle = \frac{E_a+E_b}{2}$, which matches the standard solution. The solution process includes necessary steps and shows correct reasoning, making it physically equivalent to the standard answer.

2.2.6 Model: meta-llama/Meta-Llama-3.1-70B-Instruct

Attempt 1 (2025-01-31T11:04:16.113398Z)

Detailed Solution: To find the expectation value of the energy at time t, we first need to find the time-evolved state of the system. We can do this by solving the time-dependent Schrdinger equation.
The time-evolved state is given by $\psi(t) = e^{-iHt/\hbar}\psi(0)$, where H is the Hamiltonian and $\psi(0)$ is the initial state.
First, we need to find the eigenvalues and eigenvectors of the Hamiltonian.
latex H = $\begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$
The characteristic equation is given by $\det(H - \lambda I) = 0$, where I is the identity matrix.
latex $\det(H - \lambda I) = \det \begin{pmatrix} E_a - \lambda & 0 & A \\ 0 & E_b - \lambda & 0 \\ A & 0 & E_a - \lambda \end{pmatrix} = (E_b - \lambda)(\lambda^2 - 2E_a\lambda + E_a^2 - A^2) = 0$
The eigenvalues are $\lambda_1 = E_b$, $\lambda_2 = E_a + A$, and $\lambda_3 = E_a - A$.
The corresponding eigenvectors are
latex $\psi_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \psi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \psi_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$
We can now write the initial state as a linear combination of the eigenvectors.
latex $\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\psi_2 + \frac{1}{2}\psi_3 + \frac{1}{\sqrt{2}}\psi_1$
The time-evolved state is given by
latex $\psi(t) = e^{-iHt/\hbar}\psi(0) = \frac{1}{2}e^{-i\lambda_2 t/\hbar}\psi_2 + \frac{1}{2}e^{-i\lambda_3 t/\hbar}\psi_3 + \frac{1}{\sqrt{2}}e^{-i\lambda_1 t/\hbar}\psi_1$
The expectation value of the energy is given by
latex $\langle E \rangle = \langle \psi(t)|H|\psi(t)\rangle = \frac{1}{4}e^{-i\lambda_2 t/\hbar}e^{i\lambda_2 t/\hbar}\lambda_2 + \frac{1}{4}e^{-i\lambda_3 t/\hbar}e^{i\lambda_3 t/\hbar}\lambda_3 + \frac{1}{2}e^{-i\lambda_1 t/\hbar}e^{i\lambda_1 t/\hbar}\lambda_1 + \frac{1}{4}e^{-i\lambda_2 t/\hbar}e^{i\lambda_3 t/\hbar}\langle\psi_2|H|\psi_3\rangle + \frac{1}{4}e^{-i\lambda_3 t/\hbar}e^{i\lambda_2 t/\hbar}\langle\psi_3|H|\psi_2\rangle + \frac{1}{2\sqrt{2}}e^{-i\lambda_1 t/\hbar}e^{i\lambda_2 t/\hbar}\langle\psi_1|H|\psi_2\rangle + \frac{1}{2\sqrt{2}}e^{-i\lambda_1 t/\hbar}e^{i\lambda_3 t/\hbar}\langle\psi_1|H|\psi_3\rangle + \frac{1}{2\sqrt{2}}e^{-i\lambda_2 t/\hbar}e^{i\lambda_1 t/\hbar}\langle\psi_2|H|\psi_1\rangle + \frac{1}{2\sqrt{2}}e^{-i\lambda_3 t/\hbar}e^{i\lambda_1 t/\hbar}\langle\psi_3|H|\psi_1\rangle$
Using the fact that $\langle\psi_i|H|\psi_j\rangle = \lambda_i\delta_{ij}$, we get
latex $\langle E \rangle = \frac{1}{4}\lambda_2 + \frac{1}{4}\lambda_3 + \frac{1}{2}\lambda_1$
Substituting the values of λ_1, λ_2, and λ_3, we get
latex $\langle E \rangle = \frac{1}{4}(E_a + A) + \frac{1}{4}(E_a - A) + \frac{1}{2}E_b = \frac{1}{2}E_a + \frac{1}{2}E_b$
Now, let's implement this in Python.

```
        import numpy as np

def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    """
    Calculate the expectation value of the energy at time t.

    Args:
    A (float): The real parameter in the Hamiltonian.
    E_a (float): The energy level E_a.
    E_b (float): The energy level E_b.
    t (float): The time.

    Returns:
    float: The expectation value of the energy at time t.
    """
    # Calculate the expectation value of the energy
    expectation_value = 0.5 * E_a + 0.5 * E_b

    return expectation_value
```

You can use this function to calculate the expectation value of the energy at any time t. Note that the time t is not actually used in the calculation, because the expectation value of the energy is time-independent in this case.
Verification Results:

Auto verification result (['2025-02-18T05:50:02.532210+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:56:54.108050Z) Grade: A

Notes: The submitted solution correctly finds the eigenvalues and eigenvectors of the Hamiltonian, matches the initial state to the eigenvectors, and calculates the expectation value of the energy. The solution shows that the expectation value is time-independent and matches the standard solution: $\langle E \rangle = \frac{1}{2}(E_a + E_b)$. The approach is thorough and includes all necessary steps, demonstrating a clear understanding of the problem.

Attempt 2 (2025-01-31T11:05:29.570592Z)

Detailed Solution: To find the expectation value of the energy at time t, we first need to find the time-evolved state $\psi(t)$.
The time-evolution of the state is given by the Schrdinger equation:

$$i\hbar \frac{\partial \psi(t)}{\partial t} = H\psi(t)$$

Since the Hamiltonian H is time-independent, the solution to the Schrdinger equation is given by:

$$\psi(t) = e^{-iHt/\hbar}\psi(0)$$

We can find the matrix exponential $e^{-iHt/\hbar}$ by diagonalizing the Hamiltonian H. The eigenvalues and eigenvectors of H are:

$$E_1 = E_b, \quad E_2 = E_a + A, \quad E_3 = E_a - A$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

We can write the Hamiltonian H in the diagonal basis as:

$$H = \begin{pmatrix} E_b & 0 & 0 \\ 0 & E_a + A & 0 \\ 0 & 0 & E_a - A \end{pmatrix}$$

The matrix exponential $e^{-iHt/\hbar}$ is then:

$$e^{-iHt/\hbar} = \begin{pmatrix} e^{-iE_b t/\hbar} & 0 & 0 \\ 0 & e^{-i(E_a+A)t/\hbar} & 0 \\ 0 & 0 & e^{-i(E_a-A)t/\hbar} \end{pmatrix}$$

We can now transform back to the original basis:

$$e^{-iHt/\hbar} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i(E_a+A)t/\hbar} & 0 & e^{-i(E_a-A)t/\hbar} \\ 0 & \sqrt{2}e^{-iE_b t/\hbar} & 0 \\ e^{-i(E_a+A)t/\hbar} & 0 & -e^{-i(E_a-A)t/\hbar} \end{pmatrix}$$

The time-evolved state $\psi(t)$ is then:

$$\psi(t) = e^{-iHt/\hbar}\psi(0) = \frac{1}{2}\begin{pmatrix} e^{-i(E_a+A)t/\hbar} + e^{-i(E_a-A)t/\hbar} \\ \sqrt{2}e^{-iE_b t/\hbar} \\ e^{-i(E_a+A)t/\hbar} - e^{-i(E_a-A)t/\hbar} \end{pmatrix}$$

The expectation value of the energy at time t is:

$$\langle E \rangle = \psi^\dagger(t)H\psi(t)$$

Using the expression for $\psi(t)$, we get:

$$\langle E\rangle = \frac{1}{4}\left(e^{i(E_a+A)t/\hbar} + e^{i(E_a-A)t/\hbar} \quad \sqrt{2}e^{iE_bt/\hbar} \quad e^{i(E_a+A)t/\hbar} - e^{i(E_a-A)t/\hbar}\right) H \begin{pmatrix} e^{-i(E_a+A)t/\hbar} + e^{-i(E_a-A)t/\hbar} \\ \sqrt{2}e^{-iE_bt/\hbar} \\ e^{-i(E_a+A)t/\hbar} - e^{-i(E_a-A)t/\hbar} \end{pmatrix}$$

Simplifying the expression, we get:

$$\langle E\rangle = \frac{1}{2}(E_a + E_b) + \frac{A}{2}\cos(2At/\hbar)$$

Now, let's convert this expression into Python code:

```
        import numpy as np

def expectation_value(A: float, E_a: float, E_b: float, t: float, hbar: float = 1.0)
                                           -> float:
    """
    Calculate the expectation value of the energy at time t.

    Args:
    A (float): The coupling constant.
    E_a (float): The energy level of state a.
    E_b (float): The energy level of state b.
    t (float): The time.
    hbar (float, optional): The reduced Planck constant. Defaults to 1.0.

    Returns:
    float: The expectation value of the energy.
    """
    return 0.5 * (E_a + E_b) + 0.5 * A * np.cos(2 * A * t / hbar)

Example usage:
A = 1.0
E_a = 2.0
E_b = 3.0
t = 4.0

expectation_value_energy = expectation_value(A, E_a, E_b, t)
print("The expectation value of the energy is:", expectation_value_energy)
```

Note that we've assumed $\hbar = 1.0$ in the Python code, but you can change this value if needed.
Verification Results:

Auto verification result (['2025-02-18T05:50:02.532715+00:00']) Status: **Incorrect**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:57:00.808891Z) Grade: C
Notes: The submitted solution correctly identifies the eigenvalues and eigenvectors of the Hamiltonian and attempts to find the time-evolved state $\psi(t)$. However, it introduces a time-dependent term $\frac{A}{2}\cos(2At/\hbar)$ in the expectation value of energy, which is not present in the standard solution. The standard solution shows that the expectation value of energy is constant, $\langle E\rangle = \frac{1}{2}(E_a + E_b)$, independent of time. The submitted solution's approach to diagonalizing the Hamiltonian and computing the matrix exponential is valid, but the final expression for $\langle E\rangle$ is incorrect due to the additional time-dependent term. This indicates a misunderstanding of the problem's requirement for a time-independent expectation value.

Attempt 3 (2025-01-31T11:07:44.374907Z)

Detailed Solution: To find the expectation value of the energy at time t, we first need to find the time-evolved state $\psi(t)$.
The time-evolved state is given by $\psi(t) = e^{-iHt/\hbar}\psi(0)$.

First, we need to find the eigenvalues and eigenvectors of the Hamiltonian H. The characteristic equation is given by:

$$\det(H - EI) = 0$$

$$\begin{vmatrix} E_a - E & 0 & A \\ 0 & E_b - E & 0 \\ A & 0 & E_a - E \end{vmatrix} = 0$$

$$(E_b - E)(E^2 - 2E_aE + E_a^2 - A^2) = 0$$

The eigenvalues are E_b, $E_a + \sqrt{A^2}$, and $E_a - \sqrt{A^2}$.
The corresponding eigenvectors are:

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

We can now diagonalize the Hamiltonian:

$$H = UDU^\dagger$$

where U is the matrix of eigenvectors and D is the diagonal matrix of eigenvalues.

$$U = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \end{pmatrix}$$

$$D = \begin{pmatrix} E_b & 0 & 0 \\ 0 & E_a + \sqrt{A^2} & 0 \\ 0 & 0 & E_a - \sqrt{A^2} \end{pmatrix}$$

Now, we can find the time-evolved state:

$$\psi(t) = e^{-iHt/\hbar}\psi(0) = Ue^{-iDt/\hbar}U^\dagger\psi(0)$$

$$\psi(t) = U \begin{pmatrix} e^{-iE_bt/\hbar} & 0 & 0 \\ 0 & e^{-i(E_a+\sqrt{A^2})t/\hbar} & 0 \\ 0 & 0 & e^{-i(E_a-\sqrt{A^2})t/\hbar} \end{pmatrix} U^\dagger\psi(0)$$

$$\psi(t) = U \begin{pmatrix} e^{-iE_bt/\hbar} & 0 & 0 \\ 0 & e^{-i(E_a+\sqrt{A^2})t/\hbar} & 0 \\ 0 & 0 & e^{-i(E_a-\sqrt{A^2})t/\hbar} \end{pmatrix} U^\dagger \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\psi(t) = \frac{1}{2} \begin{pmatrix} e^{-i(E_a+\sqrt{A^2})t/\hbar} + e^{-i(E_a-\sqrt{A^2})t/\hbar} \\ e^{-iE_bt/\hbar}\sqrt{2} \\ e^{-i(E_a+\sqrt{A^2})t/\hbar} - e^{-i(E_a-\sqrt{A^2})t/\hbar} \end{pmatrix}$$

The expectation value of the energy is given by:

$$\langle E \rangle = \psi^\dagger(t)H\psi(t)$$

$$\langle E \rangle = \frac{1}{4} \begin{pmatrix} e^{i(E_a+\sqrt{A^2})t/\hbar} + e^{i(E_a-\sqrt{A^2})t/\hbar} & e^{iE_bt/\hbar}\sqrt{2} & e^{i(E_a+\sqrt{A^2})t/\hbar} - e^{i(E_a-\sqrt{A^2})t/\hbar} \end{pmatrix} H \begin{pmatrix} e^{-i(E_a+\sqrt{A^2})t/\hbar} + e^{-i(E_a-\sqrt{A^2})t/\hbar} \\ e^{-iE_bt/\hbar}\sqrt{2} \\ e^{-i(E_a+\sqrt{A^2})t/\hbar} - e^{-i(E_a-\sqrt{A^2})t/\hbar} \end{pmatrix}$$

$$\langle E \rangle = \frac{1}{4} \begin{pmatrix} e^{i(E_a+\sqrt{A^2})t/\hbar} + e^{i(E_a-\sqrt{A^2})t/\hbar} & e^{iE_bt/\hbar}\sqrt{2} & e^{i(E_a+\sqrt{A^2})t/\hbar} - e^{i(E_a-\sqrt{A^2})t/\hbar} \end{pmatrix} \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix} \begin{pmatrix} e^{-i(E_a+\sqrt{A^2})t/\hbar} + e^{-i(E_a-\sqrt{A^2})t/\hbar} \\ e^{-iE_bt/\hbar}\sqrt{2} \\ e^{-i(E_a+\sqrt{A^2})t/\hbar} - e^{-i(E_a-\sqrt{A^2})t/\hbar} \end{pmatrix}$$

$$\langle E\rangle = \frac{1}{4}\begin{pmatrix} e^{i(E_a+\sqrt{A^2})t/\hbar} + e^{i(E_a-\sqrt{A^2})t/\hbar} & e^{iE_bt/\hbar}\sqrt{2} & e^{i(E_a+\sqrt{A^2})t/\hbar} - e^{i(E_a-\sqrt{A^2})t/\hbar}\end{pmatrix}\begin{pmatrix} E_a(e^{-i(E_a+\sqrt{A^2})t/\hbar} + e^{-i(E_a-\sqrt{A^2})t/\hbar}) + A(e^{-i(E_a \\ E_be^{-iE_bt/\hbar}\sqrt{2} \\ A(e^{-i(E_a+\sqrt{A^2})t/\hbar} + e^{-i(E_a-\sqrt{A^2})t/\hbar}) + E_a(e^{-i(E_a \end{pmatrix}$$

$$\langle E\rangle = \frac{1}{4}\begin{pmatrix} e^{i(E_a+\sqrt{A^2})t/\hbar} + e^{i(E_a-\sqrt{A^2})t/\hbar} & e^{iE_bt/\hbar}\sqrt{2} & e^{i(E_a+\sqrt{A^2})t/\hbar} - e^{i(E_a-\sqrt{A^2})t/\hbar}\end{pmatrix}\begin{pmatrix} E_a(e^{-i(E_a+\sqrt{A^2})t/\hbar} + e^{-i(E_a-\sqrt{A^2})t/\hbar}) + A(e^{-i(E_a \\ E_be^{-iE_bt/\hbar}\sqrt{2} \\ A(e^{-i(E_a+\sqrt{A^2})t/\hbar} + e^{-i(E_a-\sqrt{A^2})t/\hbar}) + E_a(e^{-i(E_a \end{pmatrix}$$

$$\langle E\rangle = \frac{1}{2}E_a + \frac{1}{2}E_b\cos^2\left(\frac{\sqrt{A^2}t}{\hbar}\right) + \frac{1}{2}(E_a+\sqrt{A^2})\sin^2\left(\frac{\sqrt{A^2}t}{\hbar}\right)$$

Now, we can convert this expression into Python code:

```
        import math

def expectation_value(A: float, E_a: float, E_b: float, t: float, hbar: float = 1.0)
                                                -> float:
    """
    Calculate the expectation value of the energy at time t.

    Args:
    A (float): The coupling constant.
    E_a (float): The energy level of state a.
    E_b (float): The energy level of state b.
    t (float): The time at which to calculate the expectation value.
    hbar (float, optional): The reduced Planck constant. Defaults to 1.0.

    Returns:
    float: The expectation value of the energy at time t.
    """
    sqrt_A2 = math.sqrt(A**2)
    return 0.5*E_a + 0.5*E_b*math.cos(sqrt_A2*t/hbar)**2 + 0.5*(E_a + sqrt_A2)*math.
                                                     sin(sqrt_A2*t/hbar)**2
```

This function takes the coupling constant A, the energy levels E$_a andE_b, thetimet, andthereducedPlanckconstanthbarasinput, andre$

Verification Results:

Auto verification result (['2025-02-18T05:50:02.533164+00:00']) Status: **Incorrect**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:57:09.142237Z) Grade: C

Notes: The submitted solution attempts to find the expectation value of the energy by considering the time evolution of the state, which is a valid approach. However, it introduces unnecessary complexity by focusing on the time-dependent state and does not directly address the problem as stated in the standard solution, which is to find the expectation value of energy in a given state without time evolution. The final expression for the expectation value of energy, $\langle E\rangle = \frac{1}{2}E_a + \frac{1}{2}E_b\cos^2\left(\frac{\sqrt{A^2}t}{\hbar}\right) + \frac{1}{2}(E_a+\sqrt{A^2})\sin^2\left(\frac{\sqrt{A^2}t}{\hbar}\right)$, is not equivalent to the standard solution's result $\langle E\rangle = \frac{1}{2}(E_a+E_b)$. The submitted solution also includes a Python function, which is irrelevant to the problem's mathematical solution. Overall, the approach shows some understanding of quantum mechanics concepts but fails to align with the problem's requirements and the standard solution's simplicity.

Attempt 4 (2025-01-31T11:08:55.069381Z)

Detailed Solution: To find the expectation value of the energy at time t, we first need to find the time-evolved state $\psi(t)$.

The time-evolution of the state is given by the time-evolution operator $U(t) = e^{-iHt/\hbar}$, where H is the Hamiltonian of the system.
First, let's find the eigenvalues and eigenvectors of the Hamiltonian H.
latex H $= \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$
The characteristic equation of H is given by:
latex $\begin{vmatrix} E_a - \lambda & 0 & A \\ 0 & E_b - \lambda & 0 \\ A & 0 & E_a - \lambda \end{vmatrix} = 0$
Solving for λ, we get:
latex $\lambda_1 = E_b$
and
latex $\lambda_{2,3} = E_a \pm A$
The corresponding eigenvectors are:
latex $\psi_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
latex $\psi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$
latex $\psi_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$
Now, we can write the initial state $\psi(t = 0)$ as a linear combination of the eigenvectors:
latex $\psi(t = 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\psi_2 + \frac{1}{\sqrt{2}}\psi_3$
The time-evolved state $\psi(t)$ is given by:
latex $\psi(t) = e^{-iHt/\hbar}\psi(t = 0) = \frac{1}{\sqrt{2}}e^{-i\lambda_2 t/\hbar}\psi_2 + \frac{1}{\sqrt{2}}e^{-i\lambda_3 t/\hbar}\psi_3$
Now, we can find the expectation value of the energy at time t:
latex $\langle E \rangle = \langle \psi(t)|H|\psi(t)\rangle = \frac{1}{2}\left(e^{i\lambda_2 t/\hbar}\langle\psi_2| + e^{i\lambda_3 t/\hbar}\langle\psi_3|\right) H \left(e^{-i\lambda_2 t/\hbar}\psi_2 + e^{-i\lambda_3 t/\hbar}\psi_3\right)$
Using the fact that ψ_2 and ψ_3 are eigenvectors of H with eigenvalues λ_2 and λ_3, respectively, we get:
latex $\langle E \rangle = \frac{1}{2}\left(\lambda_2 + \lambda_3 + e^{i(\lambda_2-\lambda_3)t/\hbar}\langle\psi_2|H|\psi_3\rangle + e^{-i(\lambda_2-\lambda_3)t/\hbar}\langle\psi_3|H|\psi_2\rangle\right)$
Using the explicit forms of ψ_2 and ψ_3, we get:
latex $\langle E \rangle = \frac{1}{2}\left(E_a + A + E_a - A + e^{i2At/\hbar}\frac{1}{2}(E_a - E_a) + e^{-i2At/\hbar}\frac{1}{2}(E_a - E_a)\right)$
Simplifying, we get:
latex $\langle E \rangle = E_a + \frac{A}{2}\left(e^{i2At/\hbar} + e^{-i2At/\hbar}\right) = E_a + A\cos(2At/\hbar)$
Now, let's convert this expression into Python code:

```
        import numpy as np

def expectation_value(A: float, E_a: float, E_b: float, t: float, hbar: float = 1.0)
                                          -> float:
    """
    Calculate the expectation value of the energy at time t.

    Parameters:
    A (float): The coupling constant.
    E_a (float): The energy of the first level.
    E_b (float): The energy of the second level.
    t (float): The time.
    hbar (float, optional): The reduced Planck constant. Defaults to 1.0.

    Returns:
    float: The expectation value of the energy at time t.
    """
    return E_a + A * np.cos(2 * A * t / hbar)
```

Note that we have set the reduced Planck constant hbar to 1.0 by default, but you can change this value if

needed.
Verification Results:

Auto verification result (['2025-02-18T05:50:02.533572+00:00']) Status: **Incorrect**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:57:15.986571Z) Grade: C
Notes: The submitted solution correctly identifies the eigenvalues and eigenvectors of the Hamiltonian, which matches the standard solution. However, the approach to finding the expectation value of the energy is flawed. The standard solution shows that the expectation value is constant, given by $\langle E \rangle = \frac{1}{2}(E_a + E_b)$. The submitted solution incorrectly introduces a time-dependent term $E_a + A\cos(2At/\hbar)$, which is not consistent with the standard solution. The error arises from the incorrect assumption that the initial state is $\frac{1}{\sqrt{2}}\psi_2 + \frac{1}{\sqrt{2}}\psi_3$, leading to an incorrect calculation of the expectation value. The approach has some merit in terms of finding eigenvalues and eigenvectors, but the final result is incorrect.

Attempt 5 (2025-01-31T11:10:48.302088Z)

Detailed Solution: To find the expectation value of the energy at time t, we first need to find the time-evolved state $\psi(t)$.
The time-evolution of a state is given by the Schrdinger equation:

$$i\hbar \frac{d\psi(t)}{dt} = H\psi(t)$$

Since the Hamiltonian is time-independent, we can write the solution as:

$$\psi(t) = e^{-iHt/\hbar}\psi(0)$$

We need to find the matrix exponential $e^{-iHt/\hbar}$. To do this, we first need to find the eigenvalues and eigenvectors of the Hamiltonian.
The characteristic equation of the Hamiltonian is:

$$\det(H - EI) = 0$$

$$\begin{vmatrix} E_a - E & 0 & A \\ 0 & E_b - E & 0 \\ A & 0 & E_a - E \end{vmatrix} = 0$$

$$(E_b - E)(E^2 - 2E_aE + E_a^2 - A^2) = 0$$

The eigenvalues are:

$$E_1 = E_b$$

$$E_{2,3} = E_a \pm A$$

The corresponding eigenvectors are:

$$\psi_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\psi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\psi_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

We can now write the matrix exponential as:

$$e^{-iHt/\hbar} = e^{-iE_1t/\hbar}\psi_1\psi_1^\dagger + e^{-iE_2t/\hbar}\psi_2\psi_2^\dagger + e^{-iE_3t/\hbar}\psi_3\psi_3^\dagger$$

Substituting the eigenvalues and eigenvectors, we get:

$$e^{-iHt/\hbar} = e^{-iE_bt/\hbar}\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix} + e^{-i(E_a+A)t/\hbar}\frac{1}{2}\begin{pmatrix}1&0&1\\0&0&0\\1&0&1\end{pmatrix} + e^{-i(E_a-A)t/\hbar}\frac{1}{2}\begin{pmatrix}1&0&-1\\0&0&0\\-1&0&1\end{pmatrix}$$

We can now find the time-evolved state:

$$\psi(t) = e^{-iHt/\hbar}\psi(0)$$

$$\psi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i(E_a+A)t/\hbar} + e^{-i(E_a-A)t/\hbar}\\ e^{-iE_bt/\hbar}\\ e^{-i(E_a+A)t/\hbar} - e^{-i(E_a-A)t/\hbar}\end{pmatrix}$$

The expectation value of the energy is:

$$\langle E\rangle = \psi^\dagger(t)H\psi(t)$$

$$\langle E\rangle = \frac{1}{2}(E_a+A)e^{-i(E_a+A)t/\hbar}e^{i(E_a+A)t/\hbar} + \frac{1}{2}(E_a-A)e^{-i(E_a-A)t/\hbar}e^{i(E_a-A)t/\hbar} + \frac{1}{2}E_be^{-iE_bt/\hbar}e^{iE_bt/\hbar}$$

$$\langle E\rangle = \frac{1}{2}(E_a+A) + \frac{1}{2}(E_a-A) + \frac{1}{2}E_b$$

$$\langle E\rangle = E_a + \frac{1}{2}E_b$$

However, this solution is incorrect. The correct solution involves using the initial state to find the correct coefficients of the eigenvectors.
The initial state can be written as a linear combination of the eigenvectors:

$$\psi(0) = c_1\psi_1 + c_2\psi_2 + c_3\psi_3$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} = c_1\begin{pmatrix}0\\1\\0\end{pmatrix} + c_2\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} + c_3\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

Solving for the coefficients, we get:

$$c_1 = \frac{1}{\sqrt{2}}$$

$$c_2 = \frac{1}{2}$$

$$c_3 = \frac{1}{2}$$

The time-evolved state is:

$$\psi(t) = c_1e^{-iE_1t/\hbar}\psi_1 + c_2e^{-iE_2t/\hbar}\psi_2 + c_3e^{-iE_3t/\hbar}\psi_3$$

$$\psi(t) = \frac{1}{\sqrt{2}}e^{-iE_bt/\hbar}\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{1}{2}e^{-i(E_a+A)t/\hbar}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} + \frac{1}{2}e^{-i(E_a-A)t/\hbar}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

The expectation value of the energy is:

$$\langle E\rangle = \psi^\dagger(t)H\psi(t)$$

$$\langle E \rangle = \frac{1}{2} E_b e^{-iE_b t/\hbar} e^{iE_b t/\hbar} + \frac{1}{4}(E_a + A) e^{-i(E_a+A)t/\hbar} e^{i(E_a+A)t/\hbar} + \frac{1}{4}(E_a - A) e^{-i(E_a-A)t/\hbar} e^{i(E_a-A)t/\hbar}$$

$$\langle E \rangle = \frac{1}{2} E_b + \frac{1}{4}(E_a + A) + \frac{1}{4}(E_a - A)$$

$$\langle E \rangle = \frac{1}{2} E_b + \frac{1}{2} E_a$$

This is the correct solution.
Now, let's convert this solution into Python code:

```
        import numpy as np

def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    """
    Calculate the expectation value of the energy at time t.

    Parameters:
    A (float): The coupling constant.
    E_a (float): The energy level of the first state.
    E_b (float): The energy level of the second state.
    t (float): The time at which to calculate the expectation value.

    Returns:
    float: The expectation value of the energy at time t.
    """
    # Calculate the expectation value
    expectation_value = 0.5 * E_b + 0.5 * E_a

    return expectation_value
```

This code defines a function expectation$_v aluethattakesthecouplingconstantA, theenergylevelsE_a andE_b, andthetimetasinput, andre$

Verification Results:

Auto verification result (['2025-02-18T05:50:02.533957+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:57:21.881466Z) Grade: A

Notes: The submitted solution correctly finds the eigenvalues and eigenvectors of the Hamiltonian, and uses them to construct the time-evolved state $\psi(t)$. The solution then calculates the expectation value of the energy $\langle E \rangle$ using the correct coefficients for the initial state decomposition. The final expression for the expectation value $\langle E \rangle = \frac{1}{2}(E_a + E_b)$ matches the standard solution. The approach and reasoning are sound, and the solution is physically equivalent to the standard answer.

2.2.7 Model: Qwen/Qwen2.5-72B-Instruct

Attempt 1 (2025-01-31T17:48:12.251743Z)

Detailed Solution: Step 1: Understand the Problem
Given: - The Hamiltonian of a three-level system:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

where A is real. - The initial state of the system at $t = 0$:

$$\psi(t = 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Unknown: - The expectation value of the energy at time t.
Step 2: Devise a Plan
1. **Time Evolution of the State:** - The state of the system at time t is given by:

$$\psi(t) = e^{-iHt/\hbar}\psi(0)$$

- We need to find the matrix exponential $e^{-iHt/\hbar}$.
2. **Expectation Value of Energy:** - The expectation value of the energy at time t is:

$$\langle H \rangle(t) = \langle \psi(t)|H|\psi(t) \rangle$$

Step 3: Carry Out the Plan
Step 3.1: Time Evolution of the State
First, we need to find the eigenvalues and eigenvectors of the Hamiltonian H.
Eigenvalues: The characteristic equation is:

$$\det(H - \lambda I) = 0$$

$$\det \begin{pmatrix} E_a - \lambda & 0 & A \\ 0 & E_b - \lambda & 0 \\ A & 0 & E_a - \lambda \end{pmatrix} = 0$$

This simplifies to:

$$(E_a - \lambda)^2(E_b - \lambda) - A^2(E_b - \lambda) = 0$$
$$(E_b - \lambda)\left[(E_a - \lambda)^2 - A^2\right] = 0$$
$$(E_b - \lambda)\left[(E_a - \lambda - A)(E_a - \lambda + A)\right] = 0$$

Thus, the eigenvalues are:

$$\lambda_1 = E_b, \quad \lambda_2 = E_a + A, \quad \lambda_3 = E_a - A$$

Eigenvectors: For $\lambda_1 = E_b$:

$$(H - E_b I)\mathbf{v} = 0$$
$$\begin{pmatrix} E_a - E_b & 0 & A \\ 0 & 0 & 0 \\ A & 0 & E_a - E_b \end{pmatrix} \mathbf{v} = 0$$

This gives:

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_2 = E_a + A$:

$$(H - (E_a + A)I)\mathbf{v} = 0$$
$$\begin{pmatrix} -A & 0 & A \\ 0 & E_b - E_a - A & 0 \\ A & 0 & -A \end{pmatrix} \mathbf{v} = 0$$

This gives:

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda_3 = E_a - A$:

$$(H - (E_a - A)I)\mathbf{v} = 0$$

$$\begin{pmatrix} A & 0 & A \\ 0 & E_b - E_a + A & 0 \\ A & 0 & A \end{pmatrix} \mathbf{v} = 0$$

This gives:

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Step 3.2: Time Evolution of the State
The initial state $\psi(0)$ can be expressed in the eigenbasis:

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\psi(0) = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} \mathbf{v}_2 + \frac{1}{\sqrt{2}} \mathbf{v}_3 \right)$$

$$\psi(0) = \frac{1}{2} \mathbf{v}_2 + \frac{1}{2} \mathbf{v}_3$$

The time evolution of the state is:

$$\psi(t) = e^{-iHt/\hbar} \psi(0)$$

$$\psi(t) = \frac{1}{2} e^{-i(E_a+A)t/\hbar} \mathbf{v}_2 + \frac{1}{2} e^{-i(E_a-A)t/\hbar} \mathbf{v}_3$$

$$\psi(t) = \frac{1}{2} e^{-i(E_a+A)t/\hbar} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{2} e^{-i(E_a-A)t/\hbar} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\psi(t) = \frac{1}{2} \begin{pmatrix} e^{-i(E_a+A)t/\hbar} + e^{-i(E_a-A)t/\hbar} \\ 0 \\ e^{-i(E_a+A)t/\hbar} - e^{-i(E_a-A)t/\hbar} \end{pmatrix}$$

$$\psi(t) = \frac{1}{2} \begin{pmatrix} 2\cos\left(\frac{(E_a+A)t}{\hbar}\right) \\ 0 \\ 2i\sin\left(\frac{(E_a+A)t}{\hbar}\right) \end{pmatrix}$$

$$\psi(t) = \begin{pmatrix} \cos\left(\frac{(E_a+A)t}{\hbar}\right) \\ 0 \\ i\sin\left(\frac{(E_a+A)t}{\hbar}\right) \end{pmatrix}$$

Step 3.3: Expectation Value of Energy
The expectation value of the energy at time t is:

$$\langle H \rangle(t) = \langle \psi(t) | H | \psi(t) \rangle$$

$$\langle H \rangle(t) = \begin{pmatrix} \cos\left(\frac{(E_a+A)t}{\hbar}\right) & 0 & -i\sin\left(\frac{(E_a+A)t}{\hbar}\right) \end{pmatrix} \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix} \begin{pmatrix} \cos\left(\frac{(E_a+A)t}{\hbar}\right) \\ 0 \\ i\sin\left(\frac{(E_a+A)t}{\hbar}\right) \end{pmatrix}$$

$$\langle H \rangle(t) = \cos^2\left(\frac{(E_a+A)t}{\hbar}\right) E_a + \sin^2\left(\frac{(E_a+A)t}{\hbar}\right) E_a + A\cos\left(\frac{(E_a+A)t}{\hbar}\right) i\sin\left(\frac{(E_a+A)t}{\hbar}\right) - A\cos\left(\frac{(E_a+A)t}{\hbar}\right) i\sin\left(\frac{(E_a+A)t}{\hbar}\right)$$

$$\langle H \rangle(t) = E_a \left(\cos^2\left(\frac{(E_a+A)t}{\hbar}\right) + \sin^2\left(\frac{(E_a+A)t}{\hbar}\right) \right)$$

$$\langle H \rangle(t) = E_a$$

Final Answer:

$$\boxed{E_a}$$

Python Code

```
    def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
return E_a
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.534299+00:00']) Status: **Incorrect**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:57:29.842567Z) Grade: D
Notes: The submitted solution incorrectly calculates the expectation value of the energy. The standard solution shows that the expectation value is $\langle E \rangle = \frac{1}{2}(E_a + E_b)$, while the submitted solution concludes E_a. The approach in the submitted solution is flawed as it does not correctly decompose the initial state into the eigenbasis of the Hamiltonian and fails to account for the contributions from all eigenstates. Additionally, the time evolution and expectation value calculations are incorrect, leading to a wrong final answer. The submitted solution shows a misunderstanding of the problem and the necessary steps to solve it.

Attempt 2 (2025-01-31T17:49:55.772881Z)

Detailed Solution: Step 1: Understand the Problem
Given: - The Hamiltonian of a three-level system:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

- The initial state of the system at $t = 0$:

$$\psi(t = 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We need to find the expectation value of the energy at time t.
Step 2: Devise a Plan
1. **Time Evolution of the State**: - The state at time t is given by:

$$\psi(t) = e^{-iHt/\hbar}\psi(0)$$

- We need to find the matrix exponential $e^{-iHt/\hbar}$.
2. **Expectation Value of Energy**: - The expectation value of the energy is given by:

$$\langle H \rangle(t) = \langle \psi(t) | H | \psi(t) \rangle$$

Step 3: Carry Out the Plan
Step 3.1: Time Evolution of the State
First, we need to find the eigenvalues and eigenvectors of the Hamiltonian H.
The characteristic equation of H is:

$$\det(H - \lambda I) = 0$$

$$\det \begin{pmatrix} E_a - \lambda & 0 & A \\ 0 & E_b - \lambda & 0 \\ A & 0 & E_a - \lambda \end{pmatrix} = 0$$

This simplifies to:

$$(E_a - \lambda)^2(E_b - \lambda) - A^2(E_b - \lambda) = 0$$

$$(E_b - \lambda)\left[(E_a - \lambda)^2 - A^2\right] = 0$$

The eigenvalues are:

$$\lambda_1 = E_b, \quad \lambda_2 = E_a + A, \quad \lambda_3 = E_a - A$$

Next, we find the eigenvectors corresponding to these eigenvalues.
For $\lambda_1 = E_b$:

$$(H - E_b I)\mathbf{v}_1 = 0$$

$$\begin{pmatrix} E_a - E_b & 0 & A \\ 0 & 0 & 0 \\ A & 0 & E_a - E_b \end{pmatrix}\mathbf{v}_1 = 0$$

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_2 = E_a + A$:

$$(H - (E_a + A)I)\mathbf{v}_2 = 0$$

$$\begin{pmatrix} -A & 0 & A \\ 0 & E_b - E_a - A & 0 \\ A & 0 & -A \end{pmatrix}\mathbf{v}_2 = 0$$

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda_3 = E_a - A$:

$$(H - (E_a - A)I)\mathbf{v}_3 = 0$$

$$\begin{pmatrix} A & 0 & A \\ 0 & E_b - E_a + A & 0 \\ A & 0 & A \end{pmatrix}\mathbf{v}_3 = 0$$

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

The Hamiltonian can be diagonalized as:

$$H = P\Lambda P^{-1}$$

where

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} E_a + A & 0 & 0 \\ 0 & E_b & 0 \\ 0 & 0 & E_a - A \end{pmatrix}$$

The time evolution operator is:

$$U(t) = e^{-iHt/\hbar} = Pe^{-i\Lambda t/\hbar}P^{-1}$$

Step 3.2: Expectation Value of Energy
The initial state is:

$$\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We express $\psi(0)$ in the eigenbasis of H:

$$\psi(0) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

Solving for c_1, c_2, c_3:

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = c_1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

This gives:

$$c_1 = 1, \quad c_2 = \frac{1}{2}, \quad c_3 = \frac{1}{2}$$

Thus:

$$\psi(0) = \frac{1}{\sqrt{2}}\left(\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3\right)$$

The state at time t is:

$$\psi(t) = \frac{1}{\sqrt{2}}\left(e^{-iE_bt/\hbar}\mathbf{v}_1 + \frac{1}{2}e^{-i(E_a+A)t/\hbar}\mathbf{v}_2 + \frac{1}{2}e^{-i(E_a-A)t/\hbar}\mathbf{v}_3\right)$$

The expectation value of the energy is:

$$\langle H\rangle(t) = \langle\psi(t)|H|\psi(t)\rangle$$

Since H is diagonal in the eigenbasis:

$$\langle H\rangle(t) = \frac{1}{2}\left(E_b + \frac{1}{2}(E_a + A) + \frac{1}{2}(E_a - A)\right)$$

$$\langle H\rangle(t) = \frac{1}{2}\left(E_b + E_a\right)$$

Final Answer:

$$\boxed{\langle H\rangle(t) = \frac{E_a + E_b}{2}}$$

Python Code

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    return (E_a + E_b) / 2
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.534643+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:57:36.202929Z) Grade: A
Notes: The submitted solution correctly finds the eigenvalues and eigenvectors of the Hamiltonian, decomposes the initial state in terms of these eigenvectors, and calculates the expectation value of the energy. The final expression for the expectation value, $\langle H\rangle(t) = \frac{E_a+E_b}{2}$, matches the standard solution's result, $\langle E\rangle = \frac{1}{2}(E_a + E_b)$. The approach is thorough and includes all necessary steps, demonstrating a clear understanding of the problem.

Attempt 3 (2025-01-31T17:51:14.676935Z)

Detailed Solution: Step 1: Understand the Problem
Given: - The Hamiltonian of a three-level system:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

- The initial state of the system at $t = 0$:

$$\psi(t=0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We need to find the expectation value of the energy at time t.
Step 2: Devise a Plan
1. **Time Evolution of the State:** - The state of the system at time t is given by:

$$\psi(t) = e^{-iHt/\hbar}\psi(0)$$

- We need to find the time evolution operator $e^{-iHt/\hbar}$.

2. **Expectation Value of Energy:** - The expectation value of the energy at time t is:

$$\langle H \rangle(t) = \langle \psi(t) | H | \psi(t) \rangle$$

Step 3: Carry Out the Plan
Step 3.1: Diagonalize the Hamiltonian
First, we diagonalize the Hamiltonian H. The eigenvalues E_n and eigenvectors $|n\rangle$ of H are found by solving the characteristic equation:

$$\det(H - EI) = 0$$

The characteristic polynomial is:

$$\det \begin{pmatrix} E_a - E & 0 & A \\ 0 & E_b - E & 0 \\ A & 0 & E_a - E \end{pmatrix} = (E_a - E)^2 (E_b - E) - A^2 (E_b - E) = 0$$

This simplifies to:

$$(E_b - E) \left[(E_a - E)^2 - A^2 \right] = 0$$

The solutions are:

$$E_1 = E_b, \quad E_2 = E_a + A, \quad E_3 = E_a - A$$

The corresponding eigenvectors are:

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad |3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Step 3.2: Express the Initial State in the Eigenbasis
The initial state $\psi(0)$ can be expressed in the eigenbasis:

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right) + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Thus:

$$\psi(0) = \frac{1}{2} |2\rangle + \frac{1}{2} |3\rangle + \frac{1}{\sqrt{2}} |1\rangle$$

Step 3.3: Time Evolution of the State
The time evolution of the state is:

$$\psi(t) = e^{-iHt/\hbar} \psi(0) = \frac{1}{2} e^{-iE_2 t/\hbar} |2\rangle + \frac{1}{2} e^{-iE_3 t/\hbar} |3\rangle + \frac{1}{\sqrt{2}} e^{-iE_1 t/\hbar} |1\rangle$$

Step 3.4: Expectation Value of Energy
The expectation value of the energy at time t is:

$$\langle H \rangle(t) = \langle \psi(t) | H | \psi(t) \rangle$$

Since H is diagonal in the eigenbasis:

$$\langle H \rangle(t) = \frac{1}{4} E_2 + \frac{1}{4} E_3 + \frac{1}{2} E_1$$

Substituting the eigenvalues:

$$\langle H \rangle(t) = \frac{1}{4} (E_a + A) + \frac{1}{4} (E_a - A) + \frac{1}{2} E_b$$

Simplifying:

$$\langle H \rangle(t) = \frac{1}{4} (2E_a) + \frac{1}{2} E_b = \frac{1}{2} E_a + \frac{1}{2} E_b$$

Final Answer:

$$\boxed{\langle H \rangle(t) = \frac{1}{2} (E_a + E_b)}$$

Python Code

```
        def expectation_value(A: float , E_a: float , E_b: float , t: float) -> float:
    return 0.5 * (E_a + E_b)
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.534986+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:57:41.619848Z) Grade: A

Notes: The submitted solution correctly follows the steps to find the expectation value of the energy for the given Hamiltonian and initial state. The solution includes diagonalizing the Hamiltonian, expressing the initial state in the eigenbasis, and calculating the expectation value of energy. The final answer, $\langle H\rangle(t) = \frac{1}{2}(E_a + E_b)$, matches the standard solution. The approach and reasoning are sound, and the solution is physically equivalent to the standard answer.

Attempt 4 (2025-01-31T17:53:35.875132Z)

Detailed Solution: Step 1: Understand the Problem

Given: - The Hamiltonian of a three-level system:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

- The initial state of the system at $t = 0$:

$$\psi(t = 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We need to find the expectation value of the energy at time t.

Step 2: Devise a Plan

1. **Time Evolution of the State:** - The state of the system at time t is given by:

$$\psi(t) = e^{-iHt/\hbar}\psi(0)$$

- We need to find the matrix exponential $e^{-iHt/\hbar}$.

2. **Expectation Value of the Energy:** - The expectation value of the energy is given by:

$$\langle H\rangle = \langle \psi(t)|H|\psi(t)\rangle$$

Step 3: Carry Out the Plan

Step 3.1: Time Evolution of the State

First, we need to find the eigenvalues and eigenvectors of the Hamiltonian H.

The characteristic equation of H is:

$$\det(H - \lambda I) = 0$$

$$\det\begin{pmatrix} E_a - \lambda & 0 & A \\ 0 & E_b - \lambda & 0 \\ A & 0 & E_a - \lambda \end{pmatrix} = 0$$

$$(E_a - \lambda)^2(E_b - \lambda) - A^2(E_b - \lambda) = 0$$

$$(E_b - \lambda)\left[(E_a - \lambda)^2 - A^2\right] = 0$$

$$(E_b - \lambda)\left[(E_a - \lambda - A)(E_a - \lambda + A)\right] = 0$$

The eigenvalues are:

$$\lambda_1 = E_b, \quad \lambda_2 = E_a + A, \quad \lambda_3 = E_a - A$$

Next, we find the eigenvectors corresponding to these eigenvalues.

For $\lambda_1 = E_b$:

$$\begin{pmatrix} E_a - E_b & 0 & A \\ 0 & 0 & 0 \\ A & 0 & E_a - E_b \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$(E_a - E_b)x + Az = 0 \quad \text{and} \quad y \text{ is free}$$

$$x = -\frac{A}{E_a - E_b} z$$

$$\text{Eigenvector:} \quad \begin{pmatrix} -\frac{A}{E_a - E_b} \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = E_a + A$:

$$\begin{pmatrix} -A & 0 & A \\ 0 & -A & 0 \\ A & 0 & -A \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$-Ax + Az = 0 \quad \text{and} \quad y \text{ is free}$$

$$x = z$$

$$\text{Eigenvector:} \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda_3 = E_a - A$:

$$\begin{pmatrix} A & 0 & A \\ 0 & A & 0 \\ A & 0 & A \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$Ax + Az = 0 \quad \text{and} \quad y \text{ is free}$$

$$x = -z$$

$$\text{Eigenvector:} \quad \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Step 3.2: Time Evolution of the State
The initial state $\psi(0)$ can be expressed as a linear combination of the eigenvectors:

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\psi(0) = c_1 \begin{pmatrix} -\frac{A}{E_a - E_b} \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Solving for c_1, c_2, c_3:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} -\frac{A}{E_a - E_b} \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

This gives us the system of equations:

$$\frac{1}{\sqrt{2}} = -\frac{A}{E_a - E_b} c_1 + c_2 + c_3$$

$$\frac{1}{\sqrt{2}} = c_1$$

$$0 = c_1 + c_2 - c_3$$

Solving these, we get:

$$c_1 = \frac{1}{\sqrt{2}}$$

$$c_2 = \frac{1}{2\sqrt{2}} \left(1 + \frac{A}{E_a - E_b} \right)$$

$$c_3 = \frac{1}{2\sqrt{2}} \left(1 - \frac{A}{E_a - E_b} \right)$$

The state at time t is:

$$\psi(t) = c_1 e^{-iE_b t/\hbar} \begin{pmatrix} -\frac{A}{E_a-E_b} \\ 1 \\ 1 \end{pmatrix} + c_2 e^{-i(E_a+A)t/\hbar} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{-i(E_a-A)t/\hbar} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Step 3.3: Expectation Value of the Energy
The expectation value of the energy is:

$$\langle H \rangle = \langle \psi(t) | H | \psi(t) \rangle$$

Since H is diagonal in the eigenbasis, the expectation value is:

$$\langle H \rangle = |c_1|^2 E_b + |c_2|^2 (E_a + A) + |c_3|^2 (E_a - A)$$

Substituting the values of c_1, c_2, c_3:

$$|c_1|^2 = \frac{1}{2}$$

$$|c_2|^2 = \frac{1}{8} \left(1 + \frac{A}{E_a - E_b}\right)^2$$

$$|c_3|^2 = \frac{1}{8} \left(1 - \frac{A}{E_a - E_b}\right)^2$$

Thus:

$$\langle H \rangle = \frac{1}{2} E_b + \frac{1}{8} \left(1 + \frac{A}{E_a - E_b}\right)^2 (E_a + A) + \frac{1}{8} \left(1 - \frac{A}{E_a - E_b}\right)^2 (E_a - A)$$

Simplifying:

$$\langle H \rangle = \frac{1}{2} E_b + \frac{1}{8} \left((E_a + A) + \frac{2A(E_a + A)}{E_a - E_b} + \frac{A^2(E_a + A)}{(E_a - E_b)^2}\right) + \frac{1}{8} \left((E_a - A) - \frac{2A(E_a - A)}{E_a - E_b} + \frac{A^2(E_a - A)}{(E_a - E_b)^2}\right)$$

Combining terms:

$$\langle H \rangle = \frac{1}{2} E_b + \frac{1}{4} E_a + \frac{1}{4} A \left(\frac{2A}{E_a - E_b}\right)$$

Final Answer:

$$\langle H \rangle = \frac{1}{2} E_b + \frac{1}{2} E_a + \frac{A^2}{E_a - E_b}$$

Step 4: Convert to Python Code

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    return 0.5 * E_b + 0.5 * E_a + (A**2) / (E_a - E_b)
```

This function calculates the expectation value of the energy at time t for the given parameters A, E_a, and E_b.
Verification Results:

Auto verification result (['2025-02-18T05:50:02.535352+00:00']) Status: **Incorrect**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:57:48.623634Z) Grade: C
Notes: The submitted solution shows a basic understanding of the problem but has significant issues. The eigenvalues and eigenvectors are correctly calculated, but the decomposition of the initial state into eigenvectors is incorrect. The coefficients c_1, c_2, c_3 are not correctly determined, leading to an incorrect expression for the expectation value of the energy. The final answer, $\langle H \rangle = \frac{1}{2}E_b + \frac{1}{2}E_a + \frac{A^2}{E_a - E_b}$, does not match the standard solution's $\langle E \rangle = \frac{1}{2}(E_a + E_b)$. The approach has some merit, but crucial steps are flawed, leading to a wrong final result.

Attempt 5 (2025-01-31T17:55:43.782481Z)

Detailed Solution: Step 1: Understand the Problem
Given: - The Hamiltonian of a three-level system:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

- The initial state of the system at $t = 0$:

$$\psi(t=0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We need to find the expectation value of the energy at time t.
Step 2: Devise a Plan
1. **Time Evolution of the State:** - The state of the system at time t is given by:

$$\psi(t) = e^{-iHt/\hbar}\psi(0)$$

- We need to find the matrix exponential $e^{-iHt/\hbar}$.
2. **Expectation Value of Energy:** - The expectation value of the energy is given by:

$$\langle H \rangle(t) = \langle \psi(t) | H | \psi(t) \rangle$$

Step 3: Carry Out the Plan
Step 3.1: Time Evolution of the State
First, we need to find the eigenvalues and eigenvectors of the Hamiltonian H.
The characteristic equation of H is:

$$\det(H - \lambda I) = 0$$

$$\det \begin{pmatrix} E_a - \lambda & 0 & A \\ 0 & E_b - \lambda & 0 \\ A & 0 & E_a - \lambda \end{pmatrix} = 0$$

$$(E_a - \lambda)^2 (E_b - \lambda) - A^2 (E_b - \lambda) = 0$$

$$(E_b - \lambda) \left[(E_a - \lambda)^2 - A^2 \right] = 0$$

$$(E_b - \lambda) \left[(E_a - \lambda - A)(E_a - \lambda + A) \right] = 0$$

The eigenvalues are:

$$\lambda_1 = E_b, \quad \lambda_2 = E_a + A, \quad \lambda_3 = E_a - A$$

Next, we find the eigenvectors corresponding to these eigenvalues.
For $\lambda_1 = E_b$:

$$\begin{pmatrix} E_a - E_b & 0 & A \\ 0 & 0 & 0 \\ A & 0 & E_a - E_b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$(E_a - E_b)x + Az = 0, \quad y \text{ is free}, \quad Ax + (E_a - E_b)z = 0$$

$$x = 0, \quad y = 1, \quad z = 0$$

$$\text{Eigenvector: } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_2 = E_a + A$:

$$\begin{pmatrix} -A & 0 & A \\ 0 & -A & 0 \\ A & 0 & -A \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$-Ax + Az = 0, \quad -Ay = 0, \quad Ax - Az = 0$$

$$x = z, \quad y = 0$$

$$\text{Eigenvector: } \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda_3 = E_a - A$:

$$\begin{pmatrix} A & 0 & A \\ 0 & A & 0 \\ A & 0 & A \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$Ax + Az = 0, \quad Ay = 0, \quad Ax + Az = 0$$

$$x = -z, \quad y = 0$$

$$\text{Eigenvector: } \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

The Hamiltonian can be diagonalized as:

$$H = PDP^{-1}$$

where

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} E_a + A & 0 & 0 \\ 0 & E_b & 0 \\ 0 & 0 & E_a - A \end{pmatrix}$$

The time evolution operator is:

$$U(t) = e^{-iHt/\hbar} = Pe^{-iDt/\hbar}P^{-1}$$

Step 3.2: Expectation Value of Energy
The initial state is:

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We need to express $\psi(0)$ in the eigenbasis of H:

$$\psi(0) = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Solving for c_1, c_2, c_3:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} = c_1 + c_3, \quad \frac{1}{\sqrt{2}} = c_2, \quad 0 = c_1 - c_3$$

$$c_1 = c_3 = \frac{1}{2\sqrt{2}}, \quad c_2 = \frac{1}{\sqrt{2}}$$

The state at time t is:

$$\psi(t) = c_1 e^{-i(E_a+A)t/\hbar} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{-iE_b t/\hbar} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-i(E_a-A)t/\hbar} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

The expectation value of the energy is:

$$\langle H \rangle(t) = \langle \psi(t) | H | \psi(t) \rangle$$

Since H is diagonal in the eigenbasis:

$$\langle H \rangle(t) = c_1^2(E_a + A) + c_2^2 E_b + c_3^2(E_a - A)$$

Substituting c_1, c_2, c_3:

$$\langle H \rangle(t) = \left(\frac{1}{2\sqrt{2}} \right)^2 (E_a + A) + \left(\frac{1}{\sqrt{2}} \right)^2 E_b + \left(\frac{1}{2\sqrt{2}} \right)^2 (E_a - A)$$

$$\langle H \rangle(t) = \frac{1}{8}(E_a + A) + \frac{1}{2} E_b + \frac{1}{8}(E_a - A)$$

$$\langle H\rangle(t) = \frac{1}{8}E_a + \frac{1}{8}A + \frac{1}{2}E_b + \frac{1}{8}E_a - \frac{1}{8}A$$

$$\langle H\rangle(t) = \frac{1}{4}E_a + \frac{1}{2}E_b$$

Final Answer:

$$\boxed{\langle H\rangle(t) = \frac{1}{4}E_a + \frac{1}{2}E_b}$$

Python Code

```
    def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
return 0.25 * E_a + 0.5 * E_b
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.535698+00:00']) Status: **Incorrect**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:57:55.906317Z) Grade: C

Notes: The submitted solution shows a basic understanding of the problem, but there are significant issues. The eigenvectors and eigenvalues are correctly calculated, and the decomposition of the initial state in terms of the eigenvectors is attempted. However, the final expression for the expectation value of energy is incorrect. The correct expectation value should be $\langle E\rangle = \frac{1}{2}(E_a + E_b)$, as shown in the standard solution. The submitted solution incorrectly calculates $\langle H\rangle(t) = \frac{1}{4}E_a + \frac{1}{2}E_b$, which does not match the standard solution. The error likely arises from incorrect coefficients c_1, c_2, c_3 or their application in the expectation value calculation. Additionally, the time evolution aspect is not fully utilized in the final answer, which should be time-independent as per the standard solution.

2.2.8 Model: Qwen/QwQ-32B-Preview

Attempt 1 (2025-02-01T17:32:06.487145Z)

Detailed Solution: To solve this problem, we need to find the expectation value of the energy for a three-level quantum system described by a given Hamiltonian at time t, starting from a specified initial state. We will approach this problem in two main parts: first, we will use mathematical reasoning and symbolic calculations to derive a final symbolic expression for the expectation value of energy. Then, we will translate this expression into a Python function that can compute the expectation value given specific numerical inputs.
Part 1: Mathematical Solution
Step 1: Understanding the Problem
We are given the Hamiltonian of a three-level system:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

where A is real. The initial state of the system at time $t = 0$ is:

$$\psi(t = 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We need to find the expectation value of the energy at time t, which is $\langle H \rangle(t) = \langle \psi(t)|H|\psi(t)\rangle$.
Step 2: Time Evolution of the State
The state of the system at time t is given by:

$$\psi(t) = e^{-iHt/\hbar}\psi(0)$$

However, calculating the time evolution operator $e^{-iHt/\hbar}$ directly can be complex. Instead, it's often easier to find the eigenvalues and eigenvectors of H, express the initial state in terms of these eigenvectors, and then evolve each component separately.
Step 3: Diagonalizing the Hamiltonian
First, we need to find the eigenvalues and eigenvectors of H.
The eigenvalue equation is:

$$H\phi = E\phi$$

where E is the eigenvalue and ϕ is the corresponding eigenvector.
The characteristic equation is:

$$\det(H - EI) = 0$$

Calculating the determinant:

$$\det \begin{pmatrix} E_a - E & 0 & A \\ 0 & E_b - E & 0 \\ A & 0 & E_a - E \end{pmatrix} = (E_b - E)\left[(E_a - E)^2 - A^2\right] = 0$$

This gives us the eigenvalues:

$$E = E_b, \quad E = E_a + A, \quad E = E_a - A$$

Let's denote these eigenvalues as:

$$E_1 = E_b, \quad E_2 = E_a + A, \quad E_3 = E_a - A$$

Next, we find the eigenvectors corresponding to each eigenvalue.
For $E_1 = E_b$:

$$(H - E_b I)\phi_1 = 0 \implies \begin{pmatrix} E_a - E_b & 0 & A \\ 0 & 0 & 0 \\ A & 0 & E_a - E_b \end{pmatrix} \phi_1 = 0$$

This gives us the equations:

$$(E_a - E_b)\phi_{1x} + A\phi_{1z} = 0 A\phi_{1x} + (E_a - E_b)\phi_{1z} = 0$$

Assuming $E_a \neq E_b$, we can solve these equations. Lets set $\phi_{1y} = 1$ (since the second equation is $0 \cdot \phi_{1y} = 0$), and solve for ϕ_{1x} and ϕ_{1z}.
From the first equation:

$$\phi_{1z} = -\frac{E_a - E_b}{A}\phi_{1x}$$

Substituting into the third equation:

$$A\phi_{1x} + (E_a - E_b)\left(-\frac{E_a - E_b}{A}\phi_{1x}\right) = 0 A\phi_{1x} - \frac{(E_a - E_b)^2}{A}\phi_{1x} = 0 \phi_{1x}\left[A - \frac{(E_a - E_b)^2}{A}\right] = 0$$

For non-trivial solutions, $\phi_{1x} \neq 0$, so:

$$A - \frac{(E_a - E_b)^2}{A} = 0 \implies A^2 = (E_a - E_b)^2$$

But this would imply $E_a - E_b = \pm A$, which is not necessarily true. Therefore, we must have $\phi_{1x} = 0$, which implies $\phi_{1z} = 0$. Thus, the eigenvector is:

$$\phi_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

For $E_2 = E_a + A$:

$$(H - (E_a + A)I)\phi_2 = 0 \implies \begin{pmatrix} E_a - (E_a + A) & 0 & A \\ 0 & E_b - (E_a + A) & 0 \\ A & 0 & E_a - (E_a + A) \end{pmatrix}\phi_2 = 0 \implies \begin{pmatrix} -A & 0 & A \\ 0 & E_b - E_a - A & 0 \\ A & 0 & -A \end{pmatrix}\phi_2 = 0$$

This gives us the equations:

$$-A\phi_{2x} + A\phi_{2z} = 0 \implies \phi_{2z} = \phi_{2x} A\phi_{2x} - A\phi_{2z} = 0 \implies \phi_{2x} = \phi_{2z} (E_b - E_a - A)\phi_{2y} = 0$$

If $E_b \neq E_a + A$, then $\phi_{2y} = 0$. So, the eigenvector is:

$$\phi_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

For $E_3 = E_a - A$:

$$(H - (E_a - A)I)\phi_3 = 0 \implies \begin{pmatrix} E_a - (E_a - A) & 0 & A \\ 0 & E_b - (E_a - A) & 0 \\ A & 0 & E_a - (E_a - A) \end{pmatrix}\phi_3 = 0 \implies \begin{pmatrix} A & 0 & A \\ 0 & E_b - E_a + A & 0 \\ A & 0 & A \end{pmatrix}\phi_3 = 0$$

This gives us the equations:

$$A\phi_{3x} + A\phi_{3z} = 0 \implies \phi_{3z} = -\phi_{3x} A\phi_{3x} + A\phi_{3z} = 0 \implies \phi_{3x} + \phi_{3z} = 0 (E_b - E_a + A)\phi_{3y} = 0$$

If $E_b \neq E_a - A$, then $\phi_{3y} = 0$. So, the eigenvector is:

$$\phi_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Step 4: Express Initial State in Terms of Eigenvectors
The initial state is:

$$\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We need to express this state as a linear combination of the eigenvectors ϕ_1, ϕ_2, ϕ_3:

$$\psi(0) = c_1\phi_1 + c_2\phi_2 + c_3\phi_3$$

where

$$c_1 = \langle\phi_1|\psi(0)\rangle, \quad c_2 = \langle\phi_2|\psi(0)\rangle, \quad c_3 = \langle\phi_3|\psi(0)\rangle$$

Calculating the coefficients:

$$c_1 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \times 1 = \frac{1}{\sqrt{2}}$$

$$c_2 = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \times (1 \times 1 + 0 \times 1 + 1 \times 0) = \frac{1}{\sqrt{2}}$$

$$c_3 = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \times (1 \times 1 + 0 \times 1 + (-1) \times 0) = \frac{1}{\sqrt{2}}$$

Wait, there seems to be a mistake here. Let's recompute the coefficients properly, considering that the eigenvectors should be normalized.
First, normalize the eigenvectors:

$$\phi_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{already normalized}$$

$$\phi_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \|\phi_2\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}, \quad \text{normalized: } \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\phi_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \|\phi_3\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}, \quad \text{normalized: } \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Now, express $\psi(0)$ in terms of the normalized eigenvectors:

$$\psi(0) = a\phi_1 + b\phi_2 + c\phi_3$$

where

$$a = \langle\phi_1|\psi(0)\rangle = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \times 1 = \frac{1}{\sqrt{2}}$$

$$b = \langle\phi_2|\psi(0)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}(1 \times 1 + 0 \times 1 + 1 \times 0) = \frac{1}{2}$$

$$c = \langle\phi_3|\psi(0)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}(1 \times 1 + 0 \times 1 + (-1) \times 0) = \frac{1}{2}$$

Wait, there seems to be inconsistency in the coefficients. Let's double-check the normalization and the coefficients.
Given the normalized eigenvectors:

$$\phi_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \phi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \phi_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Express $\psi(0)$ in terms of these:

$$\psi(0) = a\phi_1 + b\phi_2 + c\phi_3$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} = a\begin{pmatrix}0\\1\\0\end{pmatrix} + b\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} + c\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

This gives us the system of equations:
1. $0\cdot a + \frac{b}{\sqrt{2}} + \frac{c}{\sqrt{2}} = \frac{1}{\sqrt{2}}$ (for the first component)
2. $a + 0\cdot b + 0\cdot c = \frac{1}{\sqrt{2}}$ (for the second component)
3. $0\cdot a + \frac{b}{\sqrt{2}} - \frac{c}{\sqrt{2}} = 0$ (for the third component)

From equation 2:

$$a = \frac{1}{\sqrt{2}}$$

From equation 1:

$$\frac{b+c}{\sqrt{2}} = \frac{1}{\sqrt{2}} \implies b + c = 1$$

From equation 3:

$$\frac{b-c}{\sqrt{2}} = 0 \implies b - c = 0 \implies b = c$$

Combining $b + c = 1$ and $b = c$:

$$2b = 1 \implies b = \frac{1}{2}, \quad c = \frac{1}{2}$$

Therefore, the expansion coefficients are:

$$a = \frac{1}{\sqrt{2}}, \quad b = \frac{1}{2}, \quad c = \frac{1}{2}$$

Step 5: Time Evolution of the State
The state at time t is:

$$\psi(t) = ae^{-iE_1 t/\hbar}\phi_1 + be^{-iE_2 t/\hbar}\phi_2 + ce^{-iE_3 t/\hbar}\phi_3$$

Step 6: Expectation Value of Energy
The expectation value of energy is:

$$\langle H\rangle(t) = \langle\psi(t)|H|\psi(t)\rangle$$

Since H is Hermitian, and ϕ_n are its eigenvectors, we have:

$$H\phi_n = E_n\phi_n$$

Thus,

$$\langle H\rangle(t) = \sum_n |c_n|^2 E_n + \sum_{n\neq m} c_n^* c_m e^{-i(E_n-E_m)t/\hbar}\langle\phi_n|H|\phi_m\rangle$$

However, since ϕ_n are orthogonal, $\langle\phi_n|H|\phi_m\rangle = E_m\langle\phi_n|\phi_m\rangle = E_m\delta_{nm}$, the cross terms vanish, and we get:

$$\langle H\rangle(t) = \sum_n |c_n|^2 E_n$$

Therefore, the expectation value of energy is time-independent and equals:

$$\langle H\rangle = |a|^2 E_1 + |b|^2 E_2 + |c|^2 E_3$$

Plugging in the values:

$$\langle H\rangle = \left|\frac{1}{\sqrt{2}}\right|^2 E_b + \left|\frac{1}{2}\right|^2 (E_a + A) + \left|\frac{1}{2}\right|^2 (E_a - A)$$

$$= \frac{1}{2}E_b + \frac{1}{4}(E_a + A) + \frac{1}{4}(E_a - A)$$

$$= \frac{1}{2}E_b + \frac{1}{4}E_a + \frac{1}{4}A + \frac{1}{4}E_a - \frac{1}{4}A$$

$$= \frac{1}{2}E_b + \frac{1}{2}E_a$$

So, the expectation value of energy is:

$$\boxed{\frac{E_a + E_b}{2}}$$

Part 2: Python Solution
Given that the expectation value of energy is time-independent and equals $\frac{E_a+E_b}{2}$, the Python function to compute this is straightforward.

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    return (E_a + E_b) / 2
```

Final Solution To solve this problem, we need to find the expectation value of the energy for a three-level quantum system described by a given Hamiltonian at time t, starting from a specified initial state. We will approach this problem in two main parts: first, we will use mathematical reasoning and symbolic calculations to derive a final symbolic expression for the expectation value of energy. Then, we will translate this expression into a Python function that can compute the expectation value given specific numerical inputs.
Mathematical Solution

1. **Understanding the Problem:** - Hamiltonian: $H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$ - Initial state: $\psi(t=0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ - Find $\langle H \rangle(t) = \langle \psi(t)|H|\psi(t)\rangle$
2. **Time Evolution of the State:** - Use the eigenvalues and eigenvectors of H to express the state at time t.
3. **Diagonalizing the Hamiltonian:** - Find eigenvalues: $E_1 = E_b$, $E_2 = E_a + A$, $E_3 = E_a - A$ - Find corresponding eigenvectors: - $\phi_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ - $\phi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ - $\phi_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$
4. **Express Initial State in Terms of Eigenvectors:** - Coefficients: - $a = \frac{1}{\sqrt{2}}$ - $b = \frac{1}{2}$ - $c = \frac{1}{2}$
5. **Expectation Value of Energy:** - Since the expectation value is time-independent:

$$\langle H \rangle = |a|^2 E_1 + |b|^2 E_2 + |c|^2 E_3 = \frac{1}{2}E_b + \frac{1}{4}(E_a + A) + \frac{1}{4}(E_a - A) = \frac{E_a + E_b}{2}$$

Final Answer

$$\boxed{\frac{E_a + E_b}{2}}$$

Python Solution To compute the expectation value of energy given the parameters, use the following Python function:

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    return (E_a + E_b) / 2
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.539649+00:00']) Status: **Correct**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:59:11.709649Z) Grade: A
Notes: The submitted solution correctly follows the standard solution's approach to finding the expectation value of energy for the given quantum system. It accurately identifies the eigenvalues and eigenvectors of the Hamiltonian, expresses the initial state in terms of these eigenvectors, and calculates the expectation value of energy. The final result, $\langle E \rangle = \frac{1}{2}(E_a + E_b)$, matches the standard solution. The solution process is detailed and logically structured, demonstrating a clear understanding of the problem.

Attempt 2 (2025-02-01T17:38:46.424345Z)

Detailed Solution: Alright, I've got this problem here about a three-level quantum system. The Hamiltonian is given by this matrix:

$$\text{H} = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

And the initial state at time $t = 0$ is:

$$\psi(t = 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

I need to find the expectation value of the energy at time t. Okay, let's break this down step by step.
First, I need to recall that in quantum mechanics, the expectation value of an observable is given by the inner product of the state vector with the operator representing that observable, and then with the state vector again. Since the Hamiltonian represents the energy, the expectation value of the energy is:

$$\langle H \rangle = \langle \psi(t) | H | \psi(t) \rangle$$

So, I need to find $\psi(t)$, the state vector at time t, and then compute this inner product.
But, to find $\psi(t)$, I need to know how the state evolves in time. In quantum mechanics, the time evolution of a state is governed by the Schrdinger equation:

$$\text{i d} \frac{}{dt|\psi(t)\rangle = H|\psi(t)\rangle}$$

The solution to this equation is:

$$— \psi(t)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle$$

So, I need to find the time evolution operator $U(t) = e^{-iHt/\hbar}$, and then apply it to the initial state $\psi(0)$.
But, exponentiating a matrix can be tricky. A common approach is to diagonalize the Hamiltonian matrix, because if you can write H as $H = PDP^{-1}$, where D is a diagonal matrix of eigenvalues, then:

$$\text{e}^{-iHt/\hbar} = Pe^{-iDt/\hbar} P^{-1}$$

So, first things first, I need to find the eigenvalues and eigenvectors of H.
Let's find the eigenvalues by solving the characteristic equation:

$$\det(H - \lambda I) = 0$$

So,

$$\det\begin{pmatrix} E_a - \lambda & 0 & A \\ 0 & E_b - \lambda & 0 \\ A & 0 & E_a - \lambda \end{pmatrix} = 0$$

The determinant of this matrix is:

$$(\mathrm{E}_a - \lambda)\begin{vmatrix} E_b - \lambda & 0 \\ 0 & E_a - \lambda \end{vmatrix} - 0 + A\begin{vmatrix} 0 & E_b - \lambda \\ A & 0 \end{vmatrix}$$

Simplifying:

$$(\mathrm{E}_a - \lambda)[(E_b - \lambda)(E_a - \lambda)] + A[-A(E_b - \lambda)]$$

$$= (\mathrm{E}_a - \lambda)^2(E_b - \lambda) - A^2(E_b - \lambda)$$

$$= (\mathrm{E}_b - \lambda)[(E_a - \lambda)^2 - A^2]$$

$$= (\mathrm{E}_b - \lambda)(E_a - \lambda - A)(E_a - \lambda + A)$$

Setting this equal to zero gives the eigenvalues:

$\lambda_1 = E_b$

$\lambda_2 = E_a + A$

$\lambda_3 = E_a - A$

Great, so now I have the eigenvalues. Next, I need to find the eigenvectors corresponding to each eigenvalue.
Let's start with $\lambda_1 = E_b$.
Solve $(H - E_b I)\vec{v} = 0$:

$$\begin{pmatrix} E_a - E_b & 0 & A \\ 0 & 0 & 0 \\ A & 0 & E_a - E_b \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives us the equations:
1. $(E_a - E_b)v_1 + Av_3 = 0$
2. $0 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 = 0$ (which is always true)
3. $Av_1 + (E_a - E_b)v_3 = 0$

From equation 1 and equation 3, we can see that they are linearly dependent if $(E_a - E_b)^2 = A^2$, but assuming general case where $E_a \neq E_b \pm A$, we can solve:
From equation 1:

v$_3 = -\frac{E_a - E_b}{A} v_1$

And equation 3:

A v$_1 + (E_a - E_b)v_3 = 0$

A v$_1 + (E_a - E_b)\left(-\frac{E_a - E_b}{A} v_1\right) = 0$

A v$_1 - \frac{(E_a - E_b)^2}{A} v_1 = 0$

$\left(A - \frac{(E_a - E_b)^2}{A}\right) v_1 = 0$

This should be zero for non-trivial solutions, which implies:

A - (E$_a - E_b)^2 \overline{\scriptstyle A=0}$
A$^2 = (E_a - E_b)^2$

But this would only hold if $E_a - E_b = \pm A$, which is a special case. Assuming general case where $E_a - E_b \neq \pm A$, then v_1 can be freely chosen, and v_3 is determined by $v_3 = -\frac{E_a - E_b}{A} v_1$. Also, v_2 is free because the second equation is always zero.
So, the eigenvector corresponding to $\lambda_1 = E_b$ can be written as:

$$_1 = \begin{pmatrix} v_1 \\ v_2 \\ -\frac{E_a - E_b}{A} v_1 \end{pmatrix}$$

We can choose $v_1 = A$ and $v_2 = 0$, giving:

$$_1 = \begin{pmatrix} A \\ 0 \\ -(E_a - E_b) \end{pmatrix}$$

Alternatively, choose $v_1 = 0$ and $v_2 = 1$, giving:

$$_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

So, the eigenvector corresponding to $\lambda_1 = E_b$ is:

$$_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Wait a minute, that seems simpler. Let's verify:
If $\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then:

$$\text{H}\ _1 = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ E_b \\ 0 \end{pmatrix} = E_b \vec{v}_1$$

Yes, that checks out.
Now, for $\lambda_2 = E_a + A$, solve $(H - (E_a + A)I)\vec{v} = 0$:

$$\begin{pmatrix} E_a - (E_a + A) & 0 & A \\ 0 & E_b - (E_a + A) & 0 \\ A & 0 & E_a - (E_a + A) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Simplify:

$$\begin{pmatrix} -A & 0 & A \\ 0 & E_b - E_a - A & 0 \\ A & 0 & -A \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives:
1. $-Av_1 + Av_3 = 0 \quad v_1 = v_3$
2. $(E_b - E_a - A)v_2 = 0$
3. $Av_1 - Av_3 = 0 \quad v_1 = v_3$
From equation 2, if $E_b - E_a - A \neq 0$, then $v_2 = 0$. So, the eigenvector is:

$$_2 = \begin{pmatrix} v_1 \\ 0 \\ v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

So, choosing $v_1 = 1$, we get:

$$_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Finally, for $\lambda_3 = E_a - A$, solve $(H - (E_a - A)I)\vec{v} = 0$:

$$\begin{pmatrix} E_a - (E_a - A) & 0 & A \\ 0 & E_b - (E_a - A) & 0 \\ A & 0 & E_a - (E_a - A) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Simplify:

$$\begin{pmatrix} A & 0 & A \\ 0 & E_b - E_a + A & 0 \\ A & 0 & A \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives:
1. $Av_1 + Av_3 = 0 \quad v_1 + v_3 = 0$
2. $(E_b - E_a + A)v_2 = 0$
3. $Av_1 + Av_3 = 0 \quad v_1 + v_3 = 0$

From equation 2, if $E_b - E_a + A \neq 0$, then $v_2 = 0$. From equation 1 and 3, $v_1 = -v_3$. So, the eigenvector is:

$$_3 = \begin{pmatrix} v_1 \\ 0 \\ -v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Choosing $v_1 = 1$, we get:

$$_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Now, I have the eigenvalues and eigenvectors:

$$\lambda_1 = E_b, \quad \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = E_a + A, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_3 = E_a - A, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Next, I need to express the initial state $\psi(0)$ in terms of these eigenvectors.
Given:

$$\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

I need to write this as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$:

$$\psi(0) = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$$

So,

$$\frac{1}{\sqrt{2}\begin{pmatrix}1\\1\\0\end{pmatrix}=c_1\begin{pmatrix}0\\1\\0\end{pmatrix}+c_2\begin{pmatrix}1\\0\\1\end{pmatrix}+c_3\begin{pmatrix}1\\0\\-1\end{pmatrix}}$$

This gives the system of equations:
1. $0c_1 + c_2 + c_3 = \frac{1}{\sqrt{2}}$
2. $c_1 + 0c_2 + 0c_3 = \frac{1}{\sqrt{2}}$
3. $0c_1 + c_2 - c_3 = 0$

From equation 2, $c_1 = \frac{1}{\sqrt{2}}$.
From equation 3, $c_2 - c_3 = 0 \quad c_2 = c_3$.
Substitute into equation 1:

$$c_2 + c_3 = \frac{1}{\sqrt{2}}$$

But since $c_2 = c_3$, then:

$$2\ c_2 = \frac{1}{\sqrt{2}}$$

$$c_2 = \frac{1}{2\sqrt{2}}$$

$$c_3 = \frac{1}{2\sqrt{2}}$$

So, the coefficients are:

$$c_1 = \frac{1}{\sqrt{2}}, \quad c_2 = \frac{1}{2\sqrt{2}}, \quad c_3 = \frac{1}{2\sqrt{2}}$$

Now, the state at time t is:

$$\psi(t) = c_1 e^{-i\lambda_1 t/\hbar} \vec{v}_1 + c_2 e^{-i\lambda_2 t/\hbar} \vec{v}_2 + c_3 e^{-i\lambda_3 t/\hbar} \vec{v}_3$$

The expectation value of the energy is:

$$\langle H \rangle = \langle \psi(t) | H | \psi(t) \rangle$$

But since H is Hermitian, and the eigenvectors form an orthonormal basis, we can use the fact that:

$$\langle H \rangle = |c_1|^2 \lambda_1 + |c_2|^2 \lambda_2 + |c_3|^2 \lambda_3$$

Let's verify this. In general, for a Hamiltonian with eigenvalues λ_i and eigenvectors $\vec{v}_i$, and the state expressed as $\psi(t) = \sum_i c_i e^{-i\lambda_i t/\hbar} \vec{v}_i$, the expectation value of H is:

$$\langle H \rangle = \sum_i |c_i|^2 \lambda_i$$

So, plugging in the values:

$$\langle H \rangle = \left| \frac{1}{\sqrt{2}} \right|^2 E_b + \left| \frac{1}{2\sqrt{2}} \right|^2 (E_a + A) + \left| \frac{1}{2\sqrt{2}} \right|^2 (E_a - A)$$

Calculate each term:

$$\left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$
$$\left| \frac{1}{2\sqrt{2}} \right|^2 = \frac{1}{8}$$

So,

$$\langle H \rangle = \frac{1}{2} E_b + \frac{1}{8}(E_a + A) + \frac{1}{8}(E_a - A)$$

Simplify:

$$\langle H \rangle = \frac{1}{2}E_b + \frac{1}{8}E_a + \frac{1}{8}A + \frac{1}{8}E_a - \frac{1}{8}A$$

The A terms cancel:

$$\langle H \rangle = \frac{1}{2}E_b + \frac{1}{8}E_a + \frac{1}{8}E_a$$

Combine like terms:

$$\langle H \rangle = \frac{1}{2}E_b + \frac{1}{4}E_a$$

Wait a minute, this seems independent of time, which is unusual for an expectation value in a time-dependent state. Let me double-check my steps.
I think the mistake is in assuming that $\langle H \rangle = \sum |c_i|^2 \lambda_i$. Actually, this is only true if the eigenvectors are orthonormal, which they are in this case since H is Hermitian.
But, in a time-dependent state, the coefficients pick up phase factors, but since H is time-independent, the expectation value should still be time-independent because the state evolves coherently.
Wait, no. In general, for a time-dependent state, the expectation value of H can be time-dependent if the state is not an eigenstate of H. But in this case, since H is Hermitian and the state is a superposition of its eigenvectors, the expectation value should be time-independent.
But intuitively, if the system is in a superposition of energy eigenstates, the expectation value of energy should be constant in time. So, maybe my calculation is correct after all.
Let me re-express it:

$$\langle H \rangle = \frac{1}{2}E_b + \frac{1}{4}E_a$$

Alternatively, we can write it as:

$$\langle H \rangle = \frac{E_b}{2} + \frac{E_a}{4} + \frac{E_a}{4} = \frac{E_b}{2} + \frac{E_a}{2}$$

Wait, that can't be right. Hold on, let's re-examine the calculation.
I have:

$$\langle H \rangle = \frac{1}{2}E_b + \frac{1}{8}(E_a + A) + \frac{1}{8}(E_a - A)$$

Simplifying:

$$\langle H\rangle = \frac{1}{2}E_b + \frac{1}{8}E_a + \frac{1}{8}A + \frac{1}{8}E_a - \frac{1}{8}A$$

The A terms cancel, leaving:

$$\langle H\rangle = \frac{1}{2}E_b + \frac{1}{8}E_a + \frac{1}{8}E_a = \frac{1}{2}E_b + \frac{1}{4}E_a$$

Yes, that seems correct. So, the expectation value of energy is time-independent and equals $\frac{1}{2}E_b + \frac{1}{4}E_a$.
But let's cross-verify this by computing $\langle\psi(t)|H|\psi(t)\rangle$ directly.
First, express $\psi(t)$:

$$\psi(t) = c_1 e^{-i\lambda_1 t/\hbar}\vec{v}_1 + c_2 e^{-i\lambda_2 t/\hbar}\vec{v}_2 + c_3 e^{-i\lambda_3 t/\hbar}\vec{v}_3$$

Given that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are orthonormal, the inner product $\langle\psi(t)|H|\psi(t)\rangle$ can be computed as:

$$\langle\psi(t)|H|\psi(t)\rangle = \sum_{i,j} c_i^* c_j e^{-i(\lambda_i - \lambda_j)t/\hbar}\langle v_i|H|v_j\rangle$$

But since $H|v_j\rangle = \lambda_j|v_j\rangle$, and $\langle v_i|v_j\rangle = \delta_{ij}$, this simplifies to:

$$\langle\psi(t)|H|\psi(t)\rangle = \sum_i |c_i|^2\lambda_i$$

Which is exactly what I used earlier. So, my initial calculation seems correct.
Therefore, the expectation value of energy is:

$$\langle H\rangle = \frac{1}{2}E_b + \frac{1}{4}E_a + \frac{1}{4}E_a = \frac{1}{2}E_b + \frac{1}{2}E_a$$

Wait, that can't be right because earlier I had $\frac{1}{2}E_b + \frac{1}{4}E_a$. Let me double-check the coefficients.
Given:

$$c_1 = \frac{1}{\sqrt{2}}, \quad c_2 = \frac{1}{2\sqrt{2}}, \quad c_3 = \frac{1}{2\sqrt{2}}$$

Then:

—c$_1|^2 = \frac{1}{2}, \quad |c_2|^2 = \frac{1}{8}, \quad |c_3|^2 = \frac{1}{8}$

So,

$$\langle H\rangle = \frac{1}{2}E_b + \frac{1}{8}(E_a + A) + \frac{1}{8}(E_a - A) = \frac{1}{2}E_b + \frac{1}{8}E_a + \frac{1}{8}A + \frac{1}{8}E_a - \frac{1}{8}A = \frac{1}{2}E_b + \frac{1}{4}E_a$$

Yes, that seems correct. So, the expectation value is $\frac{1}{2}E_b + \frac{1}{4}E_a$, which is time-independent.
But, perhaps there's another way to look at this. Maybe I should compute $\langle\psi(t)|H|\psi(t)\rangle$ directly without relying on the eigen decomposition.
Let's try that.
First, find $\psi(t)$:

$$\psi(t) = e^{-iHt/\hbar}\psi(0)$$

But exponentiating H is not straightforward. Alternatively, since H is Hermitian, we can use its eigen decomposition:

$$\text{H} = \text{P D P}^{-1}, \quad \text{where } D = \begin{pmatrix} E_b & 0 & 0 \\ 0 & E_a + A & 0 \\ 0 & 0 & E_a - A \end{pmatrix}$$

And P is the matrix of eigenvectors:

$$\text{P} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

Wait, let's verify this. The eigenvectors are:

$$_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

So, the matrix P should be:

$$P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

And P^{-1} can be computed as:

$$P^{-1} = \frac{1}{\det(P)} \operatorname{adj}(P)$$

First, find $\det(P)$:

$$\det(P) = 0 \cdot (0 \cdot (-1) - 0 \cdot 1) - 1 \cdot (1 \cdot (-1) - 0 \cdot 0) + 1 \cdot (1 \cdot 0 - 0 \cdot 1) = 0 - 1(-1) + 1(0) = 1$$

So, $\det(P) = 1$. Now, find the adjugate of P:

$$\operatorname{adj}(P) = \begin{pmatrix} (0 \cdot (-1) - 0 \cdot 1) & -(1 \cdot (-1) - 0 \cdot 0) & (1 \cdot 0 - 0 \cdot 0) \\ -(0 \cdot (-1) - 1 \cdot 1) & (0 \cdot (-1) - 0 \cdot 1) & -(0 \cdot 0 - 1 \cdot 1) \\ (0 \cdot 0 - 1 \cdot 1) & -(0 \cdot (-1) - 0 \cdot 0) & (0 \cdot 0 - 1 \cdot 1) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{pmatrix}$$

Therefore,

$$P^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

Now, the time evolution operator is:

$$U(t) = e^{-iHt/\hbar} = P e^{-iDt/\hbar} P^{-1}$$

Where

$$e^{-iDt/\hbar} = \begin{pmatrix} e^{-iE_b t/\hbar} & 0 & 0 \\ 0 & e^{-i(E_a + A)t/\hbar} & 0 \\ 0 & 0 & e^{-i(E_a - A)t/\hbar} \end{pmatrix}$$

Therefore,

U(t) = P e$^{-iDt/\hbar}P^{-1}$

But this seems complicated to compute directly. Given that, it's probably easier to stick with the earlier approach of expressing $\psi(t)$ in terms of the eigenvectors.
Alternatively, perhaps there's a smarter way to compute $\langle H \rangle$ without explicitly finding $\psi(t)$.
Recall that for any state ψ, $\frac{d}{dt}\langle H \rangle = \frac{d}{dt}\langle \psi(t)|H|\psi(t)\rangle$. But since H is time-independent and Hermitian, in the Heisenberg picture, H is constant, so $\langle H \rangle$ should be constant in time.
Wait, but earlier calculations suggest that $\langle H \rangle = \frac{1}{2}E_b + \frac{1}{4}E_a$, which is time-independent, aligning with this expectation.
Therefore, the expectation value of energy is:

$\langle H \rangle = \frac{1}{2}E_b + \frac{1}{4}E_a$

But to confirm, let's consider if there's any time-dependence that I might have missed.
Actually, since the coefficients c_i are constants, and the phases $e^{-i\lambda_i t/\hbar}$ cancel out in the expectation value calculation due to the Hermitian property, the expectation value is indeed time-independent.
Therefore, the final answer is:

$$\boxed{\frac{1}{2}E_b + \frac{1}{4}E_a}$$

Final Solution To solve this problem, we need to find the expectation value of the energy for a three-level quantum system described by a given Hamiltonian matrix and initial state. We will first solve this problem using mathematical reasoning and symbolic calculations, and then convert the final expression into Python code.
Approach
1. **Understand the Hamiltonian and Initial State:** - The Hamiltonian H is given as a 3x3 matrix. - The initial state $\psi(0)$ is given as a column vector.
2. **Find Eigenvalues and Eigenvectors of the Hamiltonian:** - Solve for the eigenvalues and corresponding eigenvectors of H.
3. **Express the Initial State in Terms of Eigenvectors:** - Decompose the initial state into a linear combination of the eigenvectors.
4. **Calculate the Time-Evolution of the State:** - Use the time evolution operator to find $\psi(t)$.
5. **Compute the Expectation Value of Energy:** - Calculate $\langle H \rangle = \langle \psi(t)|H|\psi(t)\rangle$.
6. **Simplify the Expression:** - Simplify the expression to get the final expectation value.
Solution Code

```
        import sympy as sp

def expectation_value(A, E_a, E_b, t):
    # Define the Hamiltonian matrix
    H = sp.Matrix([
        [E_a, 0, A],
        [0, E_b, 0],
        [A, 0, E_a]
    ])

    # Find eigenvalues and eigenvectors
    eigenvals, eigenvecs = H.eigenvals(), H.eigenvectors()
```

```
    # Initial state
    psi0 = sp.Matrix([1/sp.sqrt(2), 1/sp.sqrt(2), 0])

    # Decompose initial state into eigenvectors
    c1 = eigenvecs[0][1][0].dot(psi0)
    c2 = eigenvecs[1][1][0].dot(psi0)
    c3 = eigenvecs[2][1][0].dot(psi0)

    # Time-evolved state
    # psi(t) = c1 * exp(-i*lambda1*t/hbar) * v1 + c2 * exp(-i*lambda2*t/hbar) * v2 +
                                             c3 * exp(-i*lambda3*t/hbar) * v3
    # Since hbar is often set to 1 in theoretical calculations, we'll assume hbar=1
                                             for simplicity
    lambda1, lambda2, lambda3 = eigenvals.keys()
    v1, v2, v3 = eigenvecs[0][1][0], eigenvecs[1][1][0], eigenvecs[2][1][0]
    psi_t = c1 * sp.exp(-sp.I * lambda1 * t) * v1 + c2 * sp.exp(-sp.I * lambda2 * t)
                                             * v2 + c3 * sp.exp(-sp.I * lambda3 * t
                                             ) * v3

    # Expectation value of H
    # <H> = <psi(t)| H |psi(t)>
    # Since H is Hermitian and eigenvectors are orthonormal, <H> = |c1|^2 * lambda1 +
                                              |c2|^2 * lambda2 + |c3|^2 * lambda3
    expectation_H = sp.Abs(c1)**2 * lambda1 + sp.Abs(c2)**2 * lambda2 + sp.Abs(c3)**2
                                              * lambda3
    expectation_H = sp.simplify(expectation_H)
    return expectation_H

Example usage:
A = 1
E_a = 2
E_b = 3
t = 4
result = expectation_value(A, E_a, E_b, t)
print(result)
```

Explanation
1. **Hamiltonian and Initial State:** - The Hamiltonian H is defined as a 3x3 matrix. - The initial state $\psi(0)$ is a column vector.
2. **Eigenvalues and Eigenvectors:** - We compute the eigenvalues and eigenvectors of H.
3. **Decompose Initial State:** - The initial state is expressed as a linear combination of the eigenvectors.
4. **Time-Evolution:** - The state at time t is calculated using the time evolution operator.
5. **Expectation Value:** - The expectation value of the Hamiltonian is computed using the formula for the inner product.
6. **Simplify:** - The expression is simplified to get the final result.
This approach ensures that the expectation value is calculated accurately and efficiently, leveraging the properties of Hermitian matrices and eigen decompositions.
Verification Results:

Auto verification result (['2025-02-18T05:50:02.542959+00:00']) Status: **Incorrect**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:59:19.188244Z) Grade: C
Notes: The submitted solution shows a basic understanding of the problem and correctly identifies the eigenvalues and eigenvectors of the Hamiltonian. However, there are significant issues in the calculation of the expectation value of the energy. The decomposition of the initial state into eigenvectors is incorrect, leading to wrong coefficients. The final expression for the expectation value, $\langle H \rangle = \frac{1}{2}E_b + \frac{1}{4}E_a$, does not match the standard solution $\langle E \rangle = \frac{1}{2}(E_a + E_b)$. The approach has merit, but crucial steps and calculations are flawed.

Attempt 3 (2025-02-01T17:42:47.621886Z)

Detailed Solution: Alright, I've got this problem here about a three-level quantum system. The Hamiltonian is given by this matrix:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

And the initial state at time $t = 0$ is:

$$\psi(t = 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

I need to find the expectation value of the energy at time t. Okay, let's break this down step by step.

First, I need to recall that in quantum mechanics, the expectation value of an observable is given by the inner product of the state vector with the operator representing that observable, and then with the state vector again. Since the Hamiltonian represents the energy, the expectation value of the energy is:

$$\langle H \rangle = \langle \psi(t) | H | \psi(t) \rangle$$

So, I need to find $\psi(t)$, the state vector at time t, and then compute this inner product.

But to find $\psi(t)$, I need to know how the state evolves in time. In quantum mechanics, the time evolution of a state is governed by the Schrdinger equation:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$$

The solution to this equation is:

$$|\psi(t)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle$$

So, I need to find the time evolution operator $U(t) = e^{-iHt/\hbar}$, and then apply it to the initial state $\psi(0)$.

But exponentiating a matrix can be tricky. A standard approach is to diagonalize the Hamiltonian matrix, if possible, because exponentiating a diagonal matrix is straightforward.

So, let's try to diagonalize H. To do that, I need to find its eigenvalues and eigenvectors.

The eigenvalues E are found by solving the characteristic equation:

$$\det(H - EI) = 0$$

Where I is the identity matrix. So, let's compute $H - EI$:

$$H - EI = \begin{pmatrix} E_a - E & 0 & A \\ 0 & E_b - E & 0 \\ A & 0 & E_a - E \end{pmatrix}$$

Now, the determinant of this matrix is:

$$\det(H - EI) = (E_a - E) \begin{vmatrix} E_b - E & 0 \\ 0 & E_a - E \end{vmatrix} - 0 + A \begin{vmatrix} 0 & E_b - E \\ A & 0 \end{vmatrix}$$

Simplifying the determinants:

$$\begin{vmatrix} E_b - E & 0 \\ 0 & E_a - E \end{vmatrix} = (E_b - E)(E_a - E)$$

And:

$$\begin{vmatrix} 0 & E_b - E \\ A & 0 \end{vmatrix} = 0 \cdot 0 - (E_b - E) \cdot A = -A(E_b - E)$$

So, plugging back in:

$$\det(H - EI) = (E_a - E)(E_b - E)(E_a - E) + A[-A(E_b - E)]$$

Simplify:

$$= (E_a - E)^2(E_b - E) - A^2(E_b - E)$$

Factor out $(E_b - E)$:

$$= (E_b - E)\left[(E_a - E)^2 - A^2\right]$$

Now, set this equal to zero to find the eigenvalues:

$$(E_b - E)\left[(E_a - E)^2 - A^2\right] = 0$$

This gives us three solutions:
1. $E_b - E = 0 \quad E = E_b$
2. $(E_a - E)^2 - A^2 = 0 \quad (E_a - E)^2 = A^2 \quad E_a - E = \pm A \quad E = E_a \pm A$

So, the eigenvalues are:

$$E_1 = E_b$$

$$E_2 = E_a + A$$

$$E_3 = E_a - A$$

Great, now I need to find the eigenvectors corresponding to these eigenvalues.
Let's start with $E_1 = E_b$.
To find the eigenvector, solve:

$$(H - E_1 I)\vec{v} = 0$$

So:

$$\begin{pmatrix} E_a - E_b & 0 & A \\ 0 & E_b - E_b & 0 \\ A & 0 & E_a - E_b \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This simplifies to:

$$\begin{cases} (E_a - E_b)v_1 + Av_3 = 0 \\ 0 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 = 0 \\ Av_1 + (E_a - E_b)v_3 = 0 \end{cases}$$

The second equation is always zero, so it doesn't give any information. From the first equation:

$$(E_a - E_b)v_1 + Av_3 = 0 \quad (1)$$

And from the third equation:

$$Av_1 + (E_a - E_b)v_3 = 0 \quad (2)$$

Let's see, equation (1) and equation (2) look similar. If I solve equation (1) for v_3:

$$v_3 = -\frac{(E_a - E_b)}{A}v_1$$

Plugging this into equation (2):

$$Av_1 + (E_a - E_b)\left(-\frac{(E_a - E_b)}{A}v_1\right) = 0$$

$$Av_1 - \frac{(E_a - E_b)^2}{A}v_1 = 0$$

$$\left(A - \frac{(E_a - E_b)^2}{A}\right)v_1 = 0$$

For this to hold for non-zero v_1, the coefficient must be zero:

$$A - \frac{(E_a - E_b)^2}{A} = 0$$

$$A^2 = (E_a - E_b)^2$$

$$A = \pm(E_a - E_b)$$

Wait, this seems odd. It suggests that for $E_1 = E_b$, the eigenvector only exists if $A = \pm(E_a - E_b)$, which isn't necessarily true based on the earlier eigenvalue derivation. Maybe I made a mistake in assuming both equations are independent.

Actually, in this case, since equation (1) and equation (2) are linearly dependent (one is a multiple of the other), we only have one independent equation. So, we can choose v_1 and v_3 related by:

$$v_3 = -\frac{(E_a - E_b)}{A} v_1$$

And v_2 is free. So, the eigenvector can be written as:

$$\vec{v}_1 = \begin{pmatrix} v_1 \\ v_2 \\ -\frac{(E_a - E_b)}{A} v_1 \end{pmatrix}$$

We can choose $v_1 = A$ and $v_2 = 0$, giving:

$$\vec{v}_1 = \begin{pmatrix} A \\ 0 \\ -(E_a - E_b) \end{pmatrix}$$

Alternatively, choose $v_1 = 0$, $v_2 = 1$, giving:

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Wait, but if $v_1 = 0$, then from equation (1):

$$A v_3 = 0$$

So, $v_3 = 0$, and v_2 can be anything. So, indeed, $\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector corresponding to $E_1 = E_b$.

Similarly, for $E_2 = E_a + A$, we solve:

$$(H - E_2 I)\vec{v} = 0$$

$$\begin{pmatrix} E_a - (E_a + A) & 0 & A \\ 0 & E_b - (E_a + A) & 0 \\ A & 0 & E_a - (E_a + A) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -A & 0 & A \\ 0 & E_b - E_a - A & 0 \\ A & 0 & -A \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives the equations:

$$-A v_1 + A v_3 = 0 \quad (3)$$

$$(E_b - E_a - A) v_2 = 0 \quad (4)$$

$$A v_1 - A v_3 = 0 \quad (5)$$

Equation (3) and equation (5) are multiples of each other, so we only have one independent equation from them:

$$-Av_1 + Av_3 = 0$$

$$v_1 = v_3$$

Equation (4) is:

$$(E_b - E_a - A)v_2 = 0$$

If $E_b - E_a - A \neq 0$, then $v_2 = 0$. So, the eigenvector is:

$$\vec{v}_2 = \begin{pmatrix} v_1 \\ 0 \\ v_1 \end{pmatrix}$$

We can choose $v_1 = 1$, giving:

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Similarly, for $E_3 = E_a - A$, we solve:

$$(H - E_3 I)\vec{v} = 0$$

$$\begin{pmatrix} E_a - (E_a - A) & 0 & A \\ 0 & E_b - (E_a - A) & 0 \\ A & 0 & E_a - (E_a - A) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} A & 0 & A \\ 0 & E_b - E_a + A & 0 \\ A & 0 & A \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives the equations:

$$Av_1 + Av_3 = 0 \quad (6)$$

$$(E_b - E_a + A)v_2 = 0 \quad (7)$$

$$Av_1 + Av_3 = 0 \quad (8)$$

Again, equations (6) and (8) are the same. So, from equation (6):

$$Av_1 + Av_3 = 0$$

$$v_1 + v_3 = 0$$

$$v_3 = -v_1$$

From equation (7):
If $E_b - E_a + A \neq 0$, then $v_2 = 0$. So, the eigenvector is:

$$\vec{v}_3 = \begin{pmatrix} v_1 \\ 0 \\ -v_1 \end{pmatrix}$$

Choose $v_1 = 1$, giving:

$$\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Now, I have the eigenvalues and eigenvectors:

$$E_1 = E_b, \quad \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$E_2 = E_a + A, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$E_3 = E_a - A, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Next, I need to express the initial state $\psi(0)$ in terms of these eigenvectors.
Given:

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Let's write $\psi(0)$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$:

$$\psi(0) = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$$

So:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

This gives the system of equations:

$$0c_1 + c_2 + c_3 = \frac{1}{\sqrt{2}} \quad \text{(from first component)}$$

$$c_1 + 0c_2 + 0c_3 = \frac{1}{\sqrt{2}} \quad \text{(from second component)}$$

$$0c_1 + c_2 - c_3 = 0 \quad \text{(from third component)}$$

From the second equation:

$$c_1 = \frac{1}{\sqrt{2}}$$

From the first equation:

$$c_2 + c_3 = \frac{1}{\sqrt{2}} \quad (9)$$

From the third equation:

$$c_2 - c_3 = 0 \quad (10)$$

Adding equations (9) and (10):

$$2c_2 = \frac{1}{\sqrt{2}}$$

$$c_2 = \frac{1}{2\sqrt{2}}$$

Subtracting equation (10) from (9):

$$2c_3 = \frac{1}{\sqrt{2}}$$

$$c_3 = \frac{1}{2\sqrt{2}}$$

So, the coefficients are:

$$c_1 = \frac{1}{\sqrt{2}}$$

$$c_2 = \frac{1}{2\sqrt{2}}$$

$$c_3 = \frac{1}{2\sqrt{2}}$$

Now, the state at time t is:

$$\psi(t) = c_1 e^{-iE_1 t/\hbar}\vec{v}_1 + c_2 e^{-iE_2 t/\hbar}\vec{v}_2 + c_3 e^{-iE_3 t/\hbar}\vec{v}_3$$

The expectation value of the energy is:

$$\langle H \rangle = \langle \psi(t)|H|\psi(t)\rangle$$

But since H is Hermitian, and the eigenvectors form an orthonormal basis, we can use the fact that:

$$\langle H \rangle = |c_1|^2 E_1 + |c_2|^2 E_2 + |c_3|^2 E_3$$

Wait, is that correct? Let me think. In general, for a system in a state that is a superposition of energy eigenstates, the expectation value of the energy is the weighted sum of the eigenvalues, with weights being the squared magnitudes of the coefficients.
Given that, yes:

$$\langle H \rangle = |c_1|^2 E_1 + |c_2|^2 E_2 + |c_3|^2 E_3$$

So, let's compute this.
First, compute the magnitudes squared:

$$|c_1|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$|c_2|^2 = \left|\frac{1}{2\sqrt{2}}\right|^2 = \frac{1}{8}$$

$$|c_3|^2 = \left|\frac{1}{2\sqrt{2}}\right|^2 = \frac{1}{8}$$

Now, plug into the expectation value formula:

$$\langle H \rangle = \frac{1}{2}E_b + \frac{1}{8}(E_a + A) + \frac{1}{8}(E_a - A)$$

Simplify:

$$\langle H \rangle = \frac{1}{2}E_b + \frac{1}{8}E_a + \frac{1}{8}A + \frac{1}{8}E_a - \frac{1}{8}A$$

The A terms cancel:

$$\langle H \rangle = \frac{1}{2}E_b + \frac{1}{8}E_a + \frac{1}{8}E_a$$

$$\langle H \rangle = \frac{1}{2}E_b + \frac{1}{4}E_a$$

So, the expectation value of the energy at any time t is:

$$\langle H \rangle = \frac{1}{4}E_a + \frac{1}{2}E_b$$

Wait a minute, this seems independent of time, which is unusual because generally, expectation values can oscillate in time unless the system is in an energy eigenstate. But in this case, it's time-independent.
Let me check if I made a mistake. The key step was assuming that the expectation value is just the sum of the eigenvalues weighted by the squared coefficients. But I recall that in quantum mechanics, for a system in a superposition of energy eigenstates, the expectation value of energy is indeed time-independent and equal to this weighted sum.
So, in this case, since the system is in a superposition of energy eigenstates, and the Hamiltonian is time-independent, the expectation value of energy does not change with time.
Therefore, the final answer is:

$$\langle H \rangle = \frac{1}{4} E_a + \frac{1}{2} E_b$$

Now, to implement this in Python, I need to define a function that takes A, E_a, E_b, and t as inputs and returns the expectation value. However, since the expectation value is time-independent, the function will ignore the time variable.
So, the Python function would be:
def expectation$_valu e(A : float, E_a : float, E_b : float, t : float) - > float :$
return 0.25 * E$_a + 0.5 * E_b$
But wait, the problem mentions that A is real, and the Hamiltonian has A in it, but in the final expression for the expectation value, A cancels out. Is that correct?
Looking back, yes, in the expression for $\langle H \rangle$, the A terms did cancel out, so A does not appear in the final result.
Therefore, the Python function is correct as written.
Final Answer:

$$\boxed{\frac{1}{4} E_a + \frac{1}{2} E_b}$$

Verification Results:

Auto verification result (['2025-02-18T05:50:02.543023+00:00']) Status: **Incorrect**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:59:25.796886Z) Grade: C
Notes: The submitted solution shows a basic understanding of the problem but contains significant issues. The eigenvalues and eigenvectors are correctly identified, and the decomposition of the initial state is attempted. However, the calculation of the expectation value of the energy is incorrect. The coefficients for the decomposition of the initial state are not correctly determined, leading to an incorrect final expression for the expectation value. The correct expectation value should be $\langle E \rangle = \frac{1}{2}(E_a + E_b)$, as shown in the standard solution. The submitted solution incorrectly calculates it as $\frac{1}{4}E_a + \frac{1}{2}E_b$.

Attempt 4 (2025-02-01T17:49:17.625694Z)

Detailed Solution: Alright, I've got this problem here about a three-level quantum system. The Hamiltonian is given by this matrix:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

And the initial state at time $t = 0$ is:

$$\psi(t = 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

I need to find the expectation value of the energy at time t. Okay, let's break this down step by step.

First, I need to recall that in quantum mechanics, the expectation value of an observable is given by the inner product of the state vector with the operator representing that observable, and then with the state vector again. Since the Hamiltonian H represents the energy, the expectation value of energy is:

$$\langle H \rangle_t = \langle \psi(t) | H | \psi(t) \rangle$$

So, I need to find $\psi(t)$, the state of the system at time t.
To find $\psi(t)$, I need to solve the time-dependent Schrdinger equation:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$$

The general solution to this equation is:

$$|\psi(t)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle$$

But to compute this, it's often easier to diagonalize H, find its eigenvalues and eigenvectors, and then express $\psi(0)$ in terms of these eigenvectors.
So, let's start by finding the eigenvalues and eigenvectors of H.
The eigenvalues E are found by solving the characteristic equation:

$$\det(H - EI) = 0$$

Where I is the identity matrix. So,

$$H - EI = \begin{pmatrix} E_a - E & 0 & A \\ 0 & E_b - E & 0 \\ A & 0 & E_a - E \end{pmatrix}$$

The determinant of this matrix is:

$$\det(H - EI) = (E_a - E) \begin{vmatrix} E_b - E & 0 \\ 0 & E_a - E \end{vmatrix} - 0 + A \begin{vmatrix} 0 & E_b - E \\ A & 0 \end{vmatrix}$$

Simplifying the determinants:

$$\begin{vmatrix} E_b - E & 0 \\ 0 & E_a - E \end{vmatrix} = (E_b - E)(E_a - E)$$

And

$$\begin{vmatrix} 0 & E_b - E \\ A & 0 \end{vmatrix} = 0 \cdot 0 - (E_b - E) \cdot A = -A(E_b - E)$$

So, plugging back in:

$$\det(H - EI) = (E_a - E)(E_b - E)(E_a - E) + A[-A(E_b - E)]$$

$$= (E_a - E)^2 (E_b - E) - A^2 (E_b - E)$$

$$= (E_b - E)\left[(E_a - E)^2 - A^2\right]$$

$$= (E_b - E)(E_a - E - A)(E_a - E + A)$$

Setting this equal to zero gives the eigenvalues:

$$E_b - E = 0 \quad \rightarrow \quad E = E_b$$

$$E_a - E - A = 0 \quad \rightarrow \quad E = E_a - A$$

$$E_a - E + A = 0 \quad \rightarrow \quad E = E_a + A$$

So, the eigenvalues are E_b, $E_a - A$, and $E_a + A$.

Next, I need to find the eigenvectors corresponding to each eigenvalue.
Let's start with $E = E_b$.
Solve $(H - E_b I)\vec{v} = 0$:

$$H - E_b I = \begin{pmatrix} E_a - E_b & 0 & A \\ 0 & 0 & 0 \\ A & 0 & E_a - E_b \end{pmatrix}$$

This gives the equations:
1. $(E_a - E_b)v_1 + Av_3 = 0$
2. $0 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 = 0$ (which is always true)
3. $Av_1 + (E_a - E_b)v_3 = 0$

From equation 1 and equation 3, we have:

$$(E_a - E_b)v_1 + Av_3 = 0$$

$$Av_1 + (E_a - E_b)v_3 = 0$$

These two equations are linearly dependent if $(E_a - E_b)^2 - A^2 = 0$, but from the characteristic equation, we know that E_b is an eigenvalue only if $E_b \neq E_a \pm A$, so generally, they are independent.
Assuming $E_b \neq E_a \pm A$, we can solve these equations.
From equation 1:

$$v_3 = -\frac{(E_a - E_b)}{A} v_1$$

Plugging into equation 3:

$$Av_1 + (E_a - E_b)\left(-\frac{(E_a - E_b)}{A} v_1\right) = 0$$

$$Av_1 - \frac{(E_a - E_b)^2}{A} v_1 = 0$$

$$\left(A - \frac{(E_a - E_b)^2}{A}\right) v_1 = 0$$

For non-trivial solutions, the coefficient must be zero, but this is not necessarily the case, so perhaps I made a mistake in assuming independence.
Alternatively, since equation 2 is always zero, we have only one independent equation, say equation 1:

$$(E_a - E_b)v_1 + Av_3 = 0$$

And v_2 is free.
So, let's set $v_2 = c$, and express v_3 in terms of v_1:

$$v_3 = -\frac{(E_a - E_b)}{A} v_1$$

Then, the eigenvector corresponding to E_b is:

$$\vec{v}_b = \begin{pmatrix} v_1 \\ c \\ -\frac{(E_a - E_b)}{A} v_1 \end{pmatrix}$$

To find a specific eigenvector, we can set $v_1 = A$, $c = 0$, then $v_3 = -(E_a - E_b)$, giving:

$$\vec{v}_b = \begin{pmatrix} A \\ 0 \\ -(E_a - E_b) \end{pmatrix}$$

Alternatively, set $v_1 = 0$, $c = 1$, then $v_3 = 0$, giving:

$$\vec{v}_b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Wait, but this seems inconsistent. Let's re-examine.
Actually, from $(H - E_b I)\vec{v} = 0$, we have:

$$\begin{pmatrix} E_a - E_b & 0 & A \\ 0 & 0 & 0 \\ A & 0 & E_a - E_b \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives:
1. $(E_a - E_b)v_1 + Av_3 = 0$
2. $0 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 = 0$
3. $Av_1 + (E_a - E_b)v_3 = 0$

Equation 2 is always satisfied. Equations 1 and 3 are:

$$(E_a - E_b)v_1 + Av_3 = 0$$

$$Av_1 + (E_a - E_b)v_3 = 0$$

If $E_a - E_b \neq 0$, these equations are linearly dependent only if $(E_a - E_b)^2 - A^2 = 0$, which is not generally true since E_b is an eigenvalue distinct from $E_a \pm A$.
Wait, but in the characteristic equation, E_b is an eigenvalue only if $E_b \neq E_a \pm A$, so $(E_a - E_b)^2 - A^2 \neq 0$.
Therefore, equations 1 and 3 are independent, and we have:
From equation 1:

$$v_3 = -\frac{(E_a - E_b)}{A} v_1$$

Plugging into equation 3:

$$Av_1 + (E_a - E_b)\left(-\frac{(E_a - E_b)}{A} v_1\right) = 0$$

$$Av_1 - \frac{(E_a - E_b)^2}{A} v_1 = 0$$

$$\left(A - \frac{(E_a - E_b)^2}{A}\right) v_1 = 0$$

For non-trivial solutions, $v_1 \neq 0$, so:

$$A - \frac{(E_a - E_b)^2}{A} = 0$$

$$A^2 - (E_a - E_b)^2 = 0$$

$$(E_a - E_b)^2 = A^2$$

$$E_a - E_b = \pm A$$

But this contradicts the assumption that E_b is an eigenvalue distinct from $E_a \pm A$. Therefore, perhaps $E_b = E_a \pm A$, but according to the characteristic equation, the eigenvalues are E_b, $E_a - A$, and $E_a + A$, which are distinct if $E_b \neq E_a \pm A$.
This suggests that E_b is a separate eigenvalue from $E_a \pm A$, and the system is diagonalizable only if these are distinct.
Wait, perhaps I need to consider that E_b is indeed distinct from $E_a \pm A$, and proceed accordingly.
Alternatively, perhaps I made a mistake in calculating the eigenvalues.
Let me recompute the characteristic equation.

$$\det(H - EI) = \begin{vmatrix} E_a - E & 0 & A \\ 0 & E_b - E & 0 \\ A & 0 & E_a - E \end{vmatrix}$$

Expanding along the second row:

$$= 0 \cdot \begin{vmatrix} 0 & A \\ 0 & E_a - E \end{vmatrix} - (E_b - E) \begin{vmatrix} E_a - E & A \\ A & E_a - E \end{vmatrix} + 0 \cdot \begin{vmatrix} E_a - E & 0 \\ A & 0 \end{vmatrix}$$

$$= -(E_b - E)\left[(E_a - E)^2 - A^2\right]$$

Setting this equal to zero:

$$-(E_b - E)\left[(E_a - E)^2 - A^2\right] = 0$$

So, the eigenvalues are:

$$E = E_b$$

$$E = E_a - A$$

$$E = E_a + A$$

Assuming $E_b \neq E_a \pm A$, these are distinct eigenvalues.
Now, let's find the eigenvectors.
For $E = E_b$:
Solve $(H - E_b I)\vec{v} = 0$:

$$H - E_b I = \begin{pmatrix} E_a - E_b & 0 & A \\ 0 & 0 & 0 \\ A & 0 & E_a - E_b \end{pmatrix}$$

This gives:
1. $(E_a - E_b)v_1 + Av_3 = 0$
2. $0 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 = 0$
3. $Av_1 + (E_a - E_b)v_3 = 0$

From equation 1 and equation 3:

$$(E_a - E_b)v_1 + Av_3 = 0$$

$$Av_1 + (E_a - E_b)v_3 = 0$$

These two equations are linearly dependent if $(E_a - E_b)^2 - A^2 = 0$, which we assumed is not the case since $E_b \neq E_a \pm A$. Therefore, we have two independent equations and one degree of freedom, which is v_2.
From equation 1:

$$v_3 = -\frac{(E_a - E_b)}{A} v_1$$

Substitute into equation 3:

$$Av_1 + (E_a - E_b)\left(-\frac{(E_a - E_b)}{A} v_1\right) = 0$$

$$Av_1 - \frac{(E_a - E_b)^2}{A} v_1 = 0$$

$$\left(A - \frac{(E_a - E_b)^2}{A}\right) v_1 = 0$$

For non-trivial solutions, $v_1 \neq 0$, so:

$$A - \frac{(E_a - E_b)^2}{A} = 0$$

$$A^2 - (E_a - E_b)^2 = 0$$

$$(E_a - E_b)^2 = A^2$$

$$E_a - E_b = \pm A$$

But this contradicts our assumption that $E_b \neq E_a \pm A$. Therefore, perhaps the only solution is $v_1 = 0$ and $v_3 = 0$, with v_2 being free.
So, the eigenvector corresponding to E_b is:

$$\vec{v}_b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Now, for $E = E_a - A$:
Solve $(H - (E_a - A)I)\vec{v} = 0$:

$$H - (E_a - A)I = \begin{pmatrix} E_a - (E_a - A) & 0 & A \\ 0 & E_b - (E_a - A) & 0 \\ A & 0 & E_a - (E_a - A) \end{pmatrix} = \begin{pmatrix} A & 0 & A \\ 0 & E_b - E_a + A & 0 \\ A & 0 & A \end{pmatrix}$$

This gives:
1. $Av_1 + 0v_2 + Av_3 = 0$
2. $0v_1 + (E_b - E_a + A)v_2 + 0v_3 = 0$
3. $Av_1 + 0v_2 + Av_3 = 0$
Equations 1 and 3 are identical:

$$Av_1 + Av_3 = 0$$

$$v_1 + v_3 = 0 \quad (\text{assuming } A \neq 0)$$

From equation 2:

$$(E_b - E_a + A)v_2 = 0$$

If $E_b \neq E_a - A$, then $E_b - E_a + A \neq 0$, so $v_2 = 0$.
Thus, the eigenvector is:

$$v_3 = -v_1$$

$$\vec{v}_{a-A} = \begin{pmatrix} v_1 \\ 0 \\ -v_1 \end{pmatrix}$$

Set $v_1 = 1$, then:

$$\vec{v}_{a-A} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Similarly, for $E = E_a + A$:
Solve $(H - (E_a + A)I)\vec{v} = 0$:

$$H - (E_a + A)I = \begin{pmatrix} E_a - (E_a + A) & 0 & A \\ 0 & E_b - (E_a + A) & 0 \\ A & 0 & E_a - (E_a + A) \end{pmatrix} = \begin{pmatrix} -A & 0 & A \\ 0 & E_b - E_a - A & 0 \\ A & 0 & -A \end{pmatrix}$$

This gives:
1. $-Av_1 + 0v_2 + Av_3 = 0$
2. $0v_1 + (E_b - E_a - A)v_2 + 0v_3 = 0$
3. $Av_1 + 0v_2 - Av_3 = 0$
Equations 1 and 3 are:

$$-Av_1 + Av_3 = 0$$

$$Av_1 - Av_3 = 0$$

Which are equivalent to $v_1 = v_3$
From equation 2:
If $E_b \neq E_a + A$, then $E_b - E_a - A \neq 0$, so $v_2 = 0$
Thus, the eigenvector is:

$$v_3 = v_1$$

$$\vec{v}_{a+A} = \begin{pmatrix} v_1 \\ 0 \\ v_1 \end{pmatrix}$$

Set $v_1 = 1$, then:

$$\vec{v}_{a+A} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Now, I have the three eigenvectors:

$$\vec{v}_b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{v}_{a-A} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\vec{v}_{a+A} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Next, I need to express the initial state $\psi(0)$ in terms of these eigenvectors.
Given:

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Let's write $\psi(0)$ as a linear combination of the eigenvectors:

$$\psi(0) = c_b \vec{v}_b + c_{a-A} \vec{v}_{a-A} + c_{a+A} \vec{v}_{a+A}$$

Plugging in the vectors:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = c_b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_{a-A} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_{a+A} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

This gives the system of equations:
1. $0c_b + c_{a-A} \cdot 1 + c_{a+A} \cdot 1 = \frac{1}{\sqrt{2}}$
2. $c_b \cdot 1 + 0 \cdot c_{a-A} + 0 \cdot c_{a+A} = \frac{1}{\sqrt{2}}$
3. $0c_b + c_{a-A} \cdot (-1) + c_{a+A} \cdot 1 = 0$

From equation 2:

$$c_b = \frac{1}{\sqrt{2}}$$

From equation 3:

$$-c_{a-A} + c_{a+A} = 0$$

$$c_{a+A} = c_{a-A}$$

From equation 1:

$$c_{a-A} + c_{a+A} = \frac{1}{\sqrt{2}}$$

$$2c_{a-A} = \frac{1}{\sqrt{2}}$$

$$c_{a-A} = \frac{1}{2\sqrt{2}}$$

$$c_{a+A} = \frac{1}{2\sqrt{2}}$$

So, the coefficients are:

$$c_b = \frac{1}{\sqrt{2}}$$

$$c_{a-A} = \frac{1}{2\sqrt{2}}$$

$$c_{a+A} = \frac{1}{2\sqrt{2}}$$

Now, the state at time t is:

$$\psi(t) = c_b e^{-iE_b t/\hbar}\vec{v}_b + c_{a-A}e^{-i(E_a-A)t/\hbar}\vec{v}_{a-A} + c_{a+A}e^{-i(E_a+A)t/\hbar}\vec{v}_{a+A}$$

The expectation value of energy is:

$$\langle H \rangle_t = \langle \psi(t)|H|\psi(t)\rangle$$

But since H is Hermitian, and the eigenvectors form an orthonormal basis, the expectation value simplifies to:

$$\langle H \rangle_t = |c_b|^2 E_b + |c_{a-A}|^2(E_a - A) + |c_{a+A}|^2(E_a + A)$$

Let's compute the magnitudes squared:

$$|c_b|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$|c_{a-A}|^2 = \left|\frac{1}{2\sqrt{2}}\right|^2 = \frac{1}{8}$$

$$|c_{a+A}|^2 = \left|\frac{1}{2\sqrt{2}}\right|^2 = \frac{1}{8}$$

So,

$$\langle H \rangle_t = \frac{1}{2}E_b + \frac{1}{8}(E_a - A) + \frac{1}{8}(E_a + A)$$

$$= \frac{1}{2}E_b + \frac{1}{8}E_a - \frac{1}{8}A + \frac{1}{8}E_a + \frac{1}{8}A$$

$$= \frac{1}{2}E_b + \frac{1}{4}E_a$$

Wait a minute, this seems independent of time, which can't be right. In quantum mechanics, expectation values can oscillate in time unless the system is in an eigenstate.

Let me check my assumption here. I used the fact that H is Hermitian and the eigenvectors are orthonormal to write the expectation value as a sum of the eigenvalues weighted by the squared magnitudes of the coefficients. However, this is only correct if the eigenvectors are orthonormal.

Let me verify if the eigenvectors are orthonormal.

Compute the inner products:

$$\langle \vec{v}_b|\vec{v}_b\rangle = 0^2 + 1^2 + 0^2 = 1$$

$$\langle \vec{v}_{a-A} | \vec{v}_{a-A} \rangle = 1^2 + 0^2 + (-1)^2 = 2$$

$$\langle \vec{v}_{a+A} | \vec{v}_{a+A} \rangle = 1^2 + 0^2 + 1^2 = 2$$

$$\langle \vec{v}_b | \vec{v}_{a-A} \rangle = 0 \cdot 1 + 1 \cdot 0 + 0 \cdot (-1) = 0$$

$$\langle \vec{v}_b | \vec{v}_{a+A} \rangle = 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 = 0$$

$$\langle \vec{v}_{a-A} | \vec{v}_{a+A} \rangle = 1 \cdot 1 + 0 \cdot 0 + (-1) \cdot 1 = 1 - 1 = 0$$

So, the eigenvectors are orthogonal, but not orthonormal. $\vec{v}_{a-A}$ and $\vec{v}_{a+A}$ have norms of $\sqrt{2}$, while $\vec{v}_b$ has a norm of 1.

Therefore, to form an orthonormal basis, I need to normalize the eigenvectors.

Normalize $\vec{v}_{a-A}$ and $\vec{v}_{a+A}$:

$$\vec{u}_{a-A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\vec{u}_{a+A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\vec{v}_b$ is already normalized:

$$\vec{u}_b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Now, express $\psi(0)$ in terms of the orthonormal eigenvectors $\vec{u}_b$, $\vec{u}_{a-A}$, and $\vec{u}_{a+A}$.

Let:

$$\psi(0) = a_b \vec{u}_b + a_{a-A} \vec{u}_{a-A} + a_{a+A} \vec{u}_{a+A}$$

Compute the coefficients:

$$a_b = \langle \vec{u}_b | \psi(0) \rangle = 0 \cdot \frac{1}{\sqrt{2}} + 1 \cdot \frac{1}{\sqrt{2}} + 0 \cdot 0 = \frac{1}{\sqrt{2}}$$

$$a_{a-A} = \langle \vec{u}_{a-A} | \psi(0) \rangle = \frac{1}{\sqrt{2}} \left(1 \cdot \frac{1}{\sqrt{2}} + 0 \cdot \frac{1}{\sqrt{2}} + (-1) \cdot 0 \right) = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$$

$$a_{a+A} = \langle \vec{u}_{a+A} | \psi(0) \rangle = \frac{1}{\sqrt{2}} \left(1 \cdot \frac{1}{\sqrt{2}} + 0 \cdot \frac{1}{\sqrt{2}} + 1 \cdot 0 \right) = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$$

So, the state at time t is:

$$\psi(t) = a_b e^{-iE_b t/\hbar} \vec{u}_b + a_{a-A} e^{-i(E_a - A)t/\hbar} \vec{u}_{a-A} + a_{a+A} e^{-i(E_a + A)t/\hbar} \vec{u}_{a+A}$$

Now, the expectation value of energy is:

$$\langle H \rangle_t = |a_b|^2 E_b + |a_{a-A}|^2 (E_a - A) + |a_{a+A}|^2 (E_a + A)$$

Compute the magnitudes squared:

$$|a_b|^2 = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$

$$|a_{a-A}|^2 = \left| \frac{1}{2} \right|^2 = \frac{1}{4}$$

$$|a_{a+A}|^2 = \left| \frac{1}{2} \right|^2 = \frac{1}{4}$$

So,

$$\langle H\rangle_t = \frac{1}{2}E_b + \frac{1}{4}(E_a - A) + \frac{1}{4}(E_a + A)$$

$$= \frac{1}{2}E_b + \frac{1}{4}E_a - \frac{1}{4}A + \frac{1}{4}E_a + \frac{1}{4}A$$

$$= \frac{1}{2}E_b + \frac{1}{2}E_a$$

Again, this seems independent of time, which can't be right. Let me check another approach.
Alternatively, perhaps I should use the time-dependent state and compute the expectation value directly.
Given that:

$$\psi(t) = e^{-iHt/\hbar}\psi(0)$$

Then,

$$\langle H\rangle_t = \langle\psi(t)|H|\psi(t)\rangle = \langle\psi(0)|e^{iHt/\hbar}He^{-iHt/\hbar}|\psi(0)\rangle$$

But since H is time-independent, $e^{iHt/\hbar}He^{-iHt/\hbar} = H$, so:

$$\langle H\rangle_t = \langle\psi(0)|H|\psi(0)\rangle$$

Wait, but this would imply that the expectation value of energy is time-independent, which is true because H is Hermitian and conserved. So, in this case, the expectation value of energy does not change with time.
Let me compute $\langle H\rangle_{t=0}$:

$$\langle H\rangle_{t=0} = \langle\psi(0)|H|\psi(0)\rangle$$

Given:

$$\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$H = \begin{pmatrix}E_a & 0 & A\\ 0 & E_b & 0\\ A & 0 & E_a\end{pmatrix}$$

First, compute $H|\psi(0)\rangle$:

$$H\psi(0) = \begin{pmatrix}E_a & 0 & A\\ 0 & E_b & 0\\ A & 0 & E_a\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}E_a\cdot 1 + 0\cdot 1 + A\cdot 0\\ 0\cdot 1 + E_b\cdot 1 + 0\cdot 0\\ A\cdot 1 + 0\cdot 1 + E_a\cdot 0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}E_a\\E_b\\A\end{pmatrix}$$

Then,

$$\langle\psi(0)|H|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 & 0\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}E_a\\E_b\\A\end{pmatrix} = \frac{1}{2}(1\cdot E_a + 1\cdot E_b + 0\cdot A) = \frac{1}{2}(E_a + E_b)$$

So, indeed, the expectation value of energy is time-independent and equal to $\frac{1}{2}(E_a + E_b)$.
Wait, but earlier I got the same result using the eigen decomposition approach. So, it seems consistent.
Therefore, the expectation value of energy at any time t is:

$$\langle H\rangle_t = \frac{1}{2}(E_a + E_b)$$

This makes sense because the Hamiltonian is Hermitian, and its expectation value is conserved in time.
So, the final answer is:

$$\boxed{\frac{1}{2}(E_a + E_b)}$$

Final Answer:

$$\boxed{\frac{1}{2}(E_a + E_b)}$$

Verification Results:

Auto verification result (['2025-02-18T05:50:02.543063+00:00']) Status: **Incorrect**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:59:33.132424Z) Grade: A

Notes: The submitted solution correctly identifies the eigenvalues and eigenvectors of the Hamiltonian matrix, and uses them to decompose the initial state. The solution then calculates the expectation value of the energy using the correct approach, showing that it is time-independent and equal to $\frac{1}{2}(E_a + E_b)$. The solution is detailed and follows a logical sequence of steps, matching the standard solution's final result. The eigenvectors are normalized correctly, and the coefficients for the state decomposition are accurately determined. Overall, the solution demonstrates a strong understanding of the quantum mechanics involved.

Attempt 5 (2025-02-01T17:55:46.223129Z)

Detailed Solution: To solve this problem, we need to find the expectation value of the energy for a three-level quantum system described by a given Hamiltonian at time t, starting from a specified initial state. We will approach this problem in two main parts: first, we will use mathematical reasoning to derive a symbolic expression for the expectation value, and second, we will translate this expression into a Python function.

Part 1: Mathematical Reasoning

Step 1: Understanding the Problem

We are given the Hamiltonian of a three-level system:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

where A is real. The initial state of the system at $t = 0$ is:

$$\psi(t = 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We need to find the expectation value of the energy at time t, which is $\langle H \rangle(t) = \langle \psi(t) | H | \psi(t) \rangle$.

Step 2: Time Evolution of the State

The state of the system at time t is given by:

$$\psi(t) = e^{-iHt/\hbar} \psi(0)$$

However, calculating this directly can be cumbersome. Instead, it's often easier to express the initial state in terms of the eigenstates of the Hamiltonian, as the time evolution of each eigenstate is simply a phase factor.

Step 3: Diagonalizing the Hamiltonian

First, we need to find the eigenvalues and eigenvectors of H.

The eigenvalues E are found by solving the characteristic equation:

$$\det(H - EI) = 0$$

where I is the identity matrix.

$$H - EI = \begin{pmatrix} E_a - E & 0 & A \\ 0 & E_b - E & 0 \\ A & 0 & E_a - E \end{pmatrix}$$

The determinant is:

$$\det(H - EI) = (E_b - E) \left[(E_a - E)^2 - A^2 \right] = 0$$

This gives us the eigenvalues:

$$E_b, \quad E_a + A, \quad E_a - A$$

Let's denote them as:

$$E_1 = E_b, \quad E_2 = E_a + A, \quad E_3 = E_a - A$$

Step 4: Finding Eigenvectors
Next, we find the eigenvectors corresponding to each eigenvalue.
For $E_1 = E_b$:

$$(H - E_b I)\vec{v}_1 = 0 \implies \begin{pmatrix} E_a - E_b & 0 & A \\ 0 & 0 & 0 \\ A & 0 & E_a - E_b \end{pmatrix} \vec{v}_1 = 0$$

This gives us the equations:

$$(E_a - E_b)v_{1x} + Av_{1z} = 0Av_{1x} + (E_a - E_b)v_{1z} = 0$$

Assuming $E_a \neq E_b$, we can solve these equations. Let's set $v_{1y} = 1$ (since the second equation is $0 = 0$, v_{1y} is free), but actually, from the Hamiltonian, v_{1y} must be zero because the off-diagonal elements couple x and z, but not y. So, $v_{1y} = 0$.
Then,

$$(E_a - E_b)v_{1x} + Av_{1z} = 0Av_{1x} + (E_a - E_b)v_{1z} = 0$$

These equations are linearly dependent, so we can use one of them. Let's use the first one:

$$v_{1z} = -\frac{E_a - E_b}{A} v_{1x}$$

Let $v_{1x} = 1$, then $v_{1z} = -\frac{E_a - E_b}{A}$. So, the eigenvector is:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -\frac{E_a - E_b}{A} \end{pmatrix}$$

For $E_2 = E_a + A$:

$$(H - (E_a + A)I)\vec{v}_2 = 0 \implies \begin{pmatrix} -E_a - A & 0 & A \\ 0 & E_b - E_a - A & 0 \\ A & 0 & -E_a - A \end{pmatrix} \vec{v}_2 = 0$$

This gives us the equations:

$$(-E_a - A)v_{2x} + Av_{2z} = 0(E_b - E_a - A)v_{2y} = 0Av_{2x} + (-E_a - A)v_{2z} = 0$$

From the second equation, if $E_b \neq E_a + A$, then $v_{2y} = 0$. From the first and third equations:

$$(-E_a - A)v_{2x} + Av_{2z} = 0Av_{2x} + (-E_a - A)v_{2z} = 0$$

These equations are linearly dependent. Using the first equation:

$$v_{2z} = \frac{E_a + A}{A} v_{2x}$$

Let $v_{2x} = 1$, then $v_{2z} = \frac{E_a + A}{A}$. So, the eigenvector is:

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ \frac{E_a + A}{A} \end{pmatrix}$$

For $E_3 = E_a - A$:

$$(H - (E_a - A)I)\vec{v}_3 = 0 \implies \begin{pmatrix} E_a - (E_a - A) & 0 & A \\ 0 & E_b - (E_a - A) & 0 \\ A & 0 & E_a - (E_a - A) \end{pmatrix} \vec{v}_3 = 0$$

Simplifying:

$$\begin{pmatrix} A & 0 & A \\ 0 & E_b - E_a + A & 0 \\ A & 0 & A \end{pmatrix} \vec{v}_3 = 0$$

This gives us the equations:

$$Av_{3x} + Av_{3z} = 0(E_b - E_a + A)v_{3y} = 0Av_{3x} + Av_{3z} = 0$$

From the first and third equations, they are the same:

$$v_{3z} = -v_{3x}$$

From the second equation, if $E_b \neq E_a - A$, then $v_{3y} = 0$. Let $v_{3x} = 1$, then $v_{3z} = -1$. So, the eigenvector is:

$$\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Step 5: Normalizing Eigenvectors
We need to normalize the eigenvectors.
For $\vec{v}_1$:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -\frac{E_a - E_b}{A} \end{pmatrix}$$

The norm is:

$$\sqrt{1^2 + 0^2 + \left(-\frac{E_a - E_b}{A}\right)^2} = \sqrt{1 + \left(\frac{E_a - E_b}{A}\right)^2} = \sqrt{\frac{A^2 + (E_a - E_b)^2}{A^2}} = \frac{\sqrt{A^2 + (E_a - E_b)^2}}{|A|}$$

So, the normalized eigenvector is:

$$\vec{u}_1 = \frac{1}{\sqrt{A^2 + (E_a - E_b)^2}} \begin{pmatrix} A \\ 0 \\ E_b - E_a \end{pmatrix}$$

For $\vec{v}_2$:

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ \frac{E_a + A}{A} \end{pmatrix}$$

The norm is:

$$\sqrt{1^2 + 0^2 + \left(\frac{E_a + A}{A}\right)^2} = \sqrt{1 + \left(\frac{E_a + A}{A}\right)^2} = \sqrt{\frac{A^2 + (E_a + A)^2}{A^2}} = \frac{\sqrt{A^2 + (E_a + A)^2}}{|A|}$$

So, the normalized eigenvector is:

$$\vec{u}_2 = \frac{1}{\sqrt{A^2 + (E_a + A)^2}} \begin{pmatrix} A \\ 0 \\ E_a + A \end{pmatrix}$$

For $\vec{v}_3$:

$$\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

The norm is:

$$\sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$$

So, the normalized eigenvector is:

$$\vec{u}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Step 6: Expanding the Initial State in Terms of Eigenvectors
The initial state is:

$$\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We need to express this as a linear combination of the normalized eigenvectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$:

$$\psi(0) = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3$$

The coefficients c_i are given by:

$$c_i = \langle u_i|\psi(0)\rangle$$

Let's compute each c_i.
For c_1:

$$c_1 = \vec{u}_1^\dagger \cdot \psi(0) = \frac{1}{\sqrt{A^2 + (E_a - E_b)^2}}\begin{pmatrix} A & 0 & E_b - E_a \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}\sqrt{A^2 + (E_a - E_b)^2}}(A \cdot 1 + 0 \cdot 1 + (E_b - E_a) \cdot 0) = \frac{A}{\sqrt{2}\sqrt{A^2 + (E_a - E_b)^2}}$$

For c_2:

$$c_2 = \vec{u}_2^\dagger \cdot \psi(0) = \frac{1}{\sqrt{A^2 + (E_a + A)^2}}\begin{pmatrix} A & 0 & E_a + A \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}\sqrt{A^2 + (E_a + A)^2}}(A \cdot 1 + 0 \cdot 1 + (E_a + A) \cdot 0) = \frac{A}{\sqrt{2}\sqrt{A^2 + (E_a + A)^2}}$$

For c_3:

$$c_3 = \vec{u}_3^\dagger \cdot \psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}(1 \cdot 1 + 0 \cdot 1 + (-1) \cdot 0) = \frac{1}{2}$$

Step 7: Time Evolution of the State
The state at time t is:

$$\psi(t) = c_1 e^{-iE_1 t/\hbar}\vec{u}_1 + c_2 e^{-iE_2 t/\hbar}\vec{u}_2 + c_3 e^{-iE_3 t/\hbar}\vec{u}_3$$

Step 8: Expectation Value of Energy
The expectation value of the energy at time t is:

$$\langle H\rangle(t) = \langle\psi(t)|H|\psi(t)\rangle$$

Since H is Hermitian, it can be expressed in terms of its eigenvalues and eigenvectors:

$$H = \sum_{i=1}^{3} E_i|u_i\rangle\langle u_i|$$

Thus,

$$\langle H\rangle(t) = \sum_{i=1}^{3} E_i|c_i|^2$$

because the time-dependent phases cancel out in the inner product.
So,

$$\langle H\rangle(t) = E_1|c_1|^2 + E_2|c_2|^2 + E_3|c_3|^2$$

Step 9: Calculating $|c_i|^2$

For $|c_1|^2$:

$$|c_1|^2 = \left(\frac{A}{\sqrt{2}\sqrt{A^2 + (E_a - E_b)^2}}\right)^2 = \frac{A^2}{2(A^2 + (E_a - E_b)^2)}$$

For $|c_2|^2$:

$$|c_2|^2 = \left(\frac{A}{\sqrt{2}\sqrt{A^2 + (E_a + A)^2}}\right)^2 = \frac{A^2}{2(A^2 + (E_a + A)^2)}$$

For $|c_3|^2$:

$$|c_3|^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

Step 10: Substituting Back into the Expectation Value

$$\langle H \rangle(t) = E_b \cdot \frac{A^2}{2(A^2 + (E_a - E_b)^2)} + (E_a + A) \cdot \frac{A^2}{2(A^2 + (E_a + A)^2)} + (E_a - A) \cdot \frac{1}{4}$$

This is the symbolic expression for the expectation value of the energy at time t.
Part 2: Python Implementation
Now, we will implement this expression in Python. We need to define a function expectation$_value that takes A$, E_a, E_b, and t as inputs and returns the expectation value of the energy.
Since the expression does not actually depend on time t, as shown in the mathematical derivation, the function will be time-independent.

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    denominator1 = A**2 + (E_a - E_b)**2
    denominator2 = A**2 + (E_a + A)**2

    term1 = E_b * (A**2) / (2 * denominator1)
    term2 = (E_a + A) * (A**2) / (2 * denominator2)
    term3 = (E_a - A) * 0.25

    return term1 + term2 + term3
```

Final Solution To solve this problem, we need to find the expectation value of the energy for a three-level quantum system described by a given Hamiltonian at time t, starting from a specified initial state. We will approach this problem in two main parts: first, we will use mathematical reasoning to derive a symbolic expression for the expectation value, and second, we will translate this expression into a Python function.
Part 1: Mathematical Reasoning
Step 1: Understanding the Problem
We are given the Hamiltonian of a three-level system:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

where A is real. The initial state of the system at $t = 0$ is:

$$\psi(t=0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We need to find the expectation value of the energy at time t, which is $\langle H \rangle(t) = \langle \psi(t) | H | \psi(t) \rangle$.
Step 2: Time Evolution of the State
The state of the system at time t is given by:

$$\psi(t) = e^{-iHt/\hbar} \psi(0)$$

However, calculating this directly can be cumbersome. Instead, it's often easier to express the initial state in terms of the eigenstates of the Hamiltonian, as the time evolution of each eigenstate is simply a phase factor.
Step 3: Diagonalizing the Hamiltonian

First, we need to find the eigenvalues and eigenvectors of H.
The eigenvalues E are found by solving the characteristic equation:

$$\det(H - EI) = 0$$

where I is the identity matrix.

$$H - EI = \begin{pmatrix} E_a - E & 0 & A \\ 0 & E_b - E & 0 \\ A & 0 & E_a - E \end{pmatrix}$$

The determinant is:

$$\det(H - EI) = (E_b - E)\left[(E_a - E)^2 - A^2\right] = 0$$

This gives us the eigenvalues:

$$E_b, \quad E_a + A, \quad E_a - A$$

Let's denote them as:

$$E_1 = E_b, \quad E_2 = E_a + A, \quad E_3 = E_a - A$$

Step 4: Finding Eigenvectors
Next, we find the eigenvectors corresponding to each eigenvalue.
For $E_1 = E_b$:

$$(H - E_b I)\vec{v}_1 = 0 \implies \begin{pmatrix} E_a - E_b & 0 & A \\ 0 & 0 & 0 \\ A & 0 & E_a - E_b \end{pmatrix} \vec{v}_1 = 0$$

This gives us the equations:

$$(E_a - E_b)v_{1x} + Av_{1z} = 0Av_{1x} + (E_a - E_b)v_{1z} = 0$$

Assuming $E_a \neq E_b$, we can solve these equations. Let's set $v_{1y} = 1$ (since the second equation is $0 = 0$, v_{1y} is free), but actually, from the Hamiltonian, v_{1y} must be zero because the off-diagonal elements couple x and z, but not y. So, $v_{1y} = 0$.
Then,

$$v_{1z} = -\frac{E_a - E_b}{A} v_{1x}$$

Let $v_{1x} = 1$, then $v_{1z} = -\frac{E_a - E_b}{A}$. So, the eigenvector is:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -\frac{E_a - E_b}{A} \end{pmatrix}$$

For $E_2 = E_a + A$:

$$(H - (E_a + A)I)\vec{v}_2 = 0 \implies \begin{pmatrix} -E_a - A & 0 & A \\ 0 & E_b - E_a - A & 0 \\ A & 0 & -E_a - A \end{pmatrix} \vec{v}_2 = 0$$

This gives us the equations:

$$(-E_a - A)v_{2x} + Av_{2z} = 0(E_b - E_a - A)v_{2y} = 0Av_{2x} + (-E_a - A)v_{2z} = 0$$

From the second equation, if $E_b \neq E_a + A$, then $v_{2y} = 0$. From the first and third equations:

$$v_{2z} = \frac{E_a + A}{A} v_{2x}$$

Let $v_{2x} = 1$, then $v_{2z} = \frac{E_a + A}{A}$. So, the eigenvector is:

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ \frac{E_a+A}{A} \end{pmatrix}$$

For $E_3 = E_a - A$**:**

$$(H - (E_a - A)I)\vec{v}_3 = 0 \implies \begin{pmatrix} E_a - (E_a - A) & 0 & A \\ 0 & E_b - (E_a - A) & 0 \\ A & 0 & E_a - (E_a - A) \end{pmatrix} \vec{v}_3 = 0$$

Simplifying:

$$\begin{pmatrix} A & 0 & A \\ 0 & E_b - E_a + A & 0 \\ A & 0 & A \end{pmatrix} \vec{v}_3 = 0$$

This gives us the equations:

$$Av_{3x} + Av_{3z} = 0(E_b - E_a + A)v_{3y} = 0Av_{3x} + Av_{3z} = 0$$

From the first and third equations, they are the same:

$$v_{3z} = -v_{3x}$$

From the second equation, if $E_b \neq E_a - A$, then $v_{3y} = 0$. Let $v_{3x} = 1$, then $v_{3z} = -1$. So, the eigenvector is:

$$\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Step 5: Normalizing Eigenvectors
We need to normalize the eigenvectors.
For $\vec{v}_1$**:**

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -\frac{E_a-E_b}{A} \end{pmatrix}$$

The norm is:

$$\sqrt{1^2 + 0^2 + \left(-\frac{E_a - E_b}{A}\right)^2} = \sqrt{1 + \left(\frac{E_a - E_b}{A}\right)^2} = \sqrt{\frac{A^2 + (E_a - E_b)^2}{A^2}} = \frac{\sqrt{A^2 + (E_a - E_b)^2}}{|A|}$$

So, the normalized eigenvector is:

$$\vec{u}_1 = \frac{1}{\sqrt{A^2 + (E_a - E_b)^2}} \begin{pmatrix} A \\ 0 \\ E_b - E_a \end{pmatrix}$$

For $\vec{v}_2$**:**

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ \frac{E_a+A}{A} \end{pmatrix}$$

The norm is:

$$\sqrt{1^2 + 0^2 + \left(\frac{E_a + A}{A}\right)^2} = \sqrt{1 + \left(\frac{E_a + A}{A}\right)^2} = \sqrt{\frac{A^2 + (E_a + A)^2}{A^2}} = \frac{\sqrt{A^2 + (E_a + A)^2}}{|A|}$$

So, the normalized eigenvector is:

$$\vec{u}_2 = \frac{1}{\sqrt{A^2 + (E_a + A)^2}} \begin{pmatrix} A \\ 0 \\ E_a + A \end{pmatrix}$$

For $\vec{v}_3$**:**

$$\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

The norm is:

$$\sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$$

So, the normalized eigenvector is:

$$\vec{u}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Step 6: Expanding the Initial State in Terms of Eigenvectors
The initial state is:

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We need to express this as a linear combination of the normalized eigenvectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$:

$$\psi(0) = c_1 \vec{u}_1 + c_2 \vec{u}_2 + c_3 \vec{u}_3$$

The coefficients c_i are given by:

$$c_i = \langle u_i | \psi(0) \rangle$$

Let's compute each c_i.
For c_1:

$$c_1 = \vec{u}_1^\dagger \cdot \psi(0) = \frac{1}{\sqrt{A^2 + (E_a - E_b)^2}} \begin{pmatrix} A & 0 & E_b - E_a \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{A}{\sqrt{2}\sqrt{A^2 + (E_a - E_b)^2}}$$

For c_2:

$$c_2 = \vec{u}_2^\dagger \cdot \psi(0) = \frac{1}{\sqrt{A^2 + (E_a + A)^2}} \begin{pmatrix} A & 0 & E_a + A \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{A}{\sqrt{2}\sqrt{A^2 + (E_a + A)^2}}$$

For c_3:

$$c_3 = \vec{u}_3^\dagger \cdot \psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}$$

Step 7: Time Evolution of the State
The state at time t is:

$$\psi(t) = c_1 e^{-iE_1 t/\hbar} \vec{u}_1 + c_2 e^{-iE_2 t/\hbar} \vec{u}_2 + c_3 e^{-iE_3 t/\hbar} \vec{u}_3$$

Step 8: Expectation Value of Energy
The expectation value of the energy at time t is:

$$\langle H \rangle(t) = \langle \psi(t) | H | \psi(t) \rangle$$

Since H is Hermitian, it can be expressed in terms of its eigenvalues and eigenvectors:

$$H = \sum_{i=1}^{3} E_i |u_i\rangle\langle u_i|$$

Thus,

$$\langle H \rangle(t) = \sum_{i=1}^{3} E_i |c_i|^2$$

because the time-dependent phases cancel out in the inner product.
So,

$$\langle H\rangle(t) = E_1|c_1|^2 + E_2|c_2|^2 + E_3|c_3|^2$$

Step 9: Calculating $|c_i|^2$
For $|c_1|^2$:

$$|c_1|^2 = \left(\frac{A}{\sqrt{2}\sqrt{A^2 + (E_a - E_b)^2}}\right)^2 = \frac{A^2}{2(A^2 + (E_a - E_b)^2)}$$

For $|c_2|^2$:

$$|c_2|^2 = \left(\frac{A}{\sqrt{2}\sqrt{A^2 + (E_a + A)^2}}\right)^2 = \frac{A^2}{2(A^2 + (E_a + A)^2)}$$

For $|c_3|^2$:

$$|c_3|^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

Step 10: Substituting Back into the Expectation Value

$$\langle H\rangle(t) = E_b \cdot \frac{A^2}{2(A^2 + (E_a - E_b)^2)} + (E_a + A) \cdot \frac{A^2}{2(A^2 + (E_a + A)^2)} + (E_a - A) \cdot \frac{1}{4}$$

This is the symbolic expression for the expectation value of the energy at time t.
Part 2: Python Implementation
Now, we will implement this expression in Python. We need to define a function expectation$_value that takes A$, E_a, E_b, and t as inputs and returns the expectation value of the energy.
Since the expression does not actually depend on time t, as shown in the mathematical derivation, the function will be time-independent.

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    denominator1 = A**2 + (E_a - E_b)**2
    denominator2 = A**2 + (E_a + A)**2

    term1 = E_b * (A**2) / (2 * denominator1)
    term2 = (E_a + A) * (A**2) / (2 * denominator2)
    term3 = (E_a - A) * 0.25

    return term1 + term2 + term3
```

Final Answer

$$\boxed{E_b \cdot \frac{A^2}{2(A^2 + (E_a - E_b)^2)} + (E_a + A) \cdot \frac{A^2}{2(A^2 + (E_a + A)^2)} + (E_a - A) \cdot \frac{1}{4}}$$

Verification Results:

Auto verification result (['2025-02-18T05:50:02.543765+00:00']) Status: **Incorrect**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:59:39.433085Z) Grade: C
Notes: The submitted solution shows a detailed understanding of the problem, including the diagonalization of the Hamiltonian and the expansion of the initial state in terms of eigenvectors. However, the final expression for the expectation value of energy is incorrect. The standard solution shows that the expectation value should be $\langle E\rangle = \frac{1}{2}(E_a + E_b)$, while the submitted solution provides a complex expression that does not simplify to this result. The approach has merit, but there are significant errors in the calculation of the expectation value.

2.2.9 Model: meta-llama/Meta-Llama-3.1-8B-Instruct

Attempt 1 (2025-02-01T00:33:34.805505Z)

Detailed Solution: To solve this problem, we'll follow the steps outlined in the problem-solving approach.
Step 1: Understand the Problem The Hamiltonian of a three-level system is given as $H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$ where A is real. The state of the system at time $t = 0$ is $\psi(t = 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. We need to find the expectation value of the energy at time t.
Step 2: Devise a Plan To find the expectation value of the energy, we need to find the time-evolved state of the system, $\psi(t)$, and then calculate the expectation value of the Hamiltonian, $\langle \psi(t)|H|\psi(t)\rangle$.
Step 3: Carry Out the Plan The time-evolution of the state is given by the Schrdinger equation, $i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$. We'll solve this differential equation to find $\psi(t)$.
First, we need to find the eigenvalues and eigenvectors of the Hamiltonian. The characteristic equation is given by $\det(H - EI) = 0$, where I is the identity matrix.
Step 4: Find the Eigenvalues and Eigenvectors The characteristic equation is given by

$$\det \begin{pmatrix} E_a - E & 0 & A \\ 0 & E_b - E & 0 \\ A & 0 & E_a - E \end{pmatrix} = 0$$
$$(E_a - E)^2 (E_b - E) - A^2 (E_b - E) = 0$$
$$(E_a - E)(E_b - E)((E_a - E) - A^2) = 0$$

The eigenvalues are $E_1 = E_b$, $E_2 = E_a - A^2$, and $E_3 = E_a$.
Step 5: Find the Eigenvectors The eigenvectors can be found by solving the equation $(H - EI)\mathbf{v} = 0$. For $E_1 = E_b$, we get $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. For $E_2 = E_a - A^2$, we get $\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. For $E_3 = E_a$, we get $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.
Step 6: Express the Initial State in Terms of the Eigenvectors The initial state can be expressed as a linear combination of the eigenvectors: $\psi(0) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. We need to find the coefficients c_1, c_2, and c_3.
Step 7: Find the Coefficients We can find the coefficients by taking the inner product of the initial state with the eigenvectors: $c_1 = \langle \psi(0)|\mathbf{v}_1\rangle$, $c_2 = \langle \psi(0)|\mathbf{v}_2\rangle$, and $c_3 = \langle \psi(0)|\mathbf{v}_3\rangle$.
Step 8: Calculate the Expectation Value The expectation value of the energy is given by $\langle \psi(t)|H|\psi(t)\rangle = \sum_{i=1}^{3} |c_i|^2 E_i e^{-iE_i t/\hbar}$.
Step 9: Simplify the Expression We can simplify the expression by substituting the values of E_i and c_i.
Step 10: Write the Final Expression The final expression for the expectation value of the energy is

$$\begin{aligned} \langle \psi(t)|H|\psi(t)\rangle &= \frac{1}{2}E_b + \frac{1}{2}(E_a - A^2) + \frac{1}{2}E_a e^{-i(E_a - E_b)t/\hbar} \\ &= \frac{1}{2}(2E_a - A^2) + \frac{1}{2}E_b + \frac{1}{2}E_a e^{-i(E_a - E_b)t/\hbar} \end{aligned}$$

Step 11: Convert to Python Code We can convert the final expression to Python code as follows:

```
        import numpy as np

def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    """
    Calculate the expectation value of the energy at time t.

    Parameters:
    A (float): The real parameter in the Hamiltonian.
    E_a (float): The energy of the first level.
    E_b (float): The energy of the second level.
    t (float): The time at which to calculate the expectation value.

    Returns:
    float: The expectation value of the energy at time t.
```

```
    """
    return 0.5 * (2 * E_a - A**2) + 0.5 * E_b + 0.5 * E_a * np.exp(-1j * (E_a - E_b)
                                                    * t / np.hbar)
```

Note that we've used the np.hbar constant from the NumPy library to represent the reduced Planck constant.

Final Answer: The final answer is $\boxed{\frac{1}{2}(2E_a - A^2) + \frac{1}{2}E_b + \frac{1}{2}E_a e^{-i(E_a - E_b)t/\hbar}}$.

Verification Results:

Auto verification result (['2025-02-18T05:50:02.536035+00:00']) Status: **Incorrect**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:58:03.064503Z) Grade: C

Notes: The submitted solution shows a basic understanding of the problem but has significant issues. The eigenvalues calculated are incorrect; the standard solution provides eigenvalues $E_a \pm A$ and E_b, while the submitted solution lists $E_1 = E_b$, $E_2 = E_a - A^2$, and $E_3 = E_a$. The eigenvectors also differ from the standard solution. The final expression for the expectation value of energy does not match the standard solution's result of $\langle E \rangle = \frac{1}{2}(E_a + E_b)$. The approach to finding the expectation value is flawed due to incorrect eigenvalues and eigenvectors, leading to an incorrect final answer.

Attempt 2 (2025-02-01T00:34:24.859125Z)

Detailed Solution: To solve this problem, we first need to find the time-evolution of the state $\psi(t)$ using the given Hamiltonian H. The time-evolution of the state is given by the Schrdinger equation:

$$i\hbar \frac{d\psi(t)}{dt} = H\psi(t)$$

We can solve this differential equation using the given initial condition $\psi(t=0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$.

First, we need to find the eigenvalues and eigenvectors of the Hamiltonian H. The characteristic equation is given by:

$$\det(H - EI) = 0$$

where I is the identity matrix and E is the eigenvalue.
Solving the characteristic equation, we get:

$$\det\begin{pmatrix}E_a - E & 0 & A\\ 0 & E_b - E & 0\\ A & 0 & E_a - E\end{pmatrix} = 0$$

Expanding the determinant, we get:

$$(E_a - E)(E_b - E)(E_a - E) - A^2(E_b - E) = 0$$

Simplifying the equation, we get:

$$(E_a - E)^2(E_b - E) - A^2(E_b - E) = 0$$

Dividing both sides by $(E_b - E)$, we get:

$$(E_a - E)^2 - A^2 = 0$$

Solving for E, we get:

$$E = \frac{E_a \pm A}{2}$$

We also have the eigenvalue E_b.

The corresponding eigenvectors can be found by solving the equation:

$$(H - EI)v = 0$$

where v is the eigenvector.
For the eigenvalue $E = \frac{E_a + A}{2}$, we get:

$$\begin{pmatrix} \frac{E_a - A}{2} & 0 & A \\ 0 & E_b - E & 0 \\ A & 0 & \frac{E_a - A}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

Solving for v, we get:

$$v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

For the eigenvalue $E = \frac{E_a - A}{2}$, we get:

$$\begin{pmatrix} \frac{E_a + A}{2} & 0 & A \\ 0 & E_b - E & 0 \\ A & 0 & \frac{E_a + A}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

Solving for v, we get:

$$v = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For the eigenvalue $E = E_b$, we get:

$$\begin{pmatrix} E_a - E_b & 0 & A \\ 0 & 0 & 0 \\ A & 0 & E_a - E_b \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

Solving for v, we get:

$$v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Now, we can write the time-evolution of the state $\psi(t)$ as a linear combination of the eigenvectors:

$$\psi(t) = c_1 e^{-iE_1 t/\hbar} v_1 + c_2 e^{-iE_2 t/\hbar} v_2 + c_3 e^{-iE_3 t/\hbar} v_3$$

where c_1, c_2, and c_3 are the coefficients of the linear combination.

Using the initial condition $\psi(t = 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, we can find the coefficients c_1, c_2, and c_3.

We get:

$$c_1 = \frac{1}{\sqrt{2}}, \quad c_2 = 0, \quad c_3 = \frac{1}{\sqrt{2}}$$

Now, we can write the time-evolution of the state $\psi(t)$ as:

$$\psi(t) = \frac{1}{\sqrt{2}} e^{-iE_1 t/\hbar} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}} e^{-iE_3 t/\hbar} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

The expectation value of the energy at time t is given by:

$$\langle E \rangle = \langle \psi(t) | H | \psi(t) \rangle$$

Using the expression for $\psi(t)$, we get:

$$\langle E\rangle = \frac{1}{2}\left(e^{iE_1t/\hbar} + e^{iE_3t/\hbar}\right)\begin{pmatrix}1 & 0 & 1\\ 0 & E_b & 0\\ 1 & 0 & -1\end{pmatrix}\begin{pmatrix}e^{-iE_1t/\hbar} & 0 & e^{-iE_3t/\hbar}\\ 0 & 1 & 0\\ e^{-iE_1t/\hbar} & 0 & e^{-iE_3t/\hbar}\end{pmatrix}$$

Simplifying the expression, we get:

$$\langle E\rangle = \frac{1}{2}\left(E_a + E_b + A\cos\left(\frac{(E_1 - E_3)t}{\hbar}\right)\right)$$

where $E_1 = \frac{E_a+A}{2}$ and $E_3 = \frac{E_a-A}{2}$.
Now, we can write the final expression for the expectation value of the energy at time t as:

$$\langle E\rangle = \frac{1}{2}\left(E_a + E_b + A\cos\left(\frac{(E_a + A - E_a + A)t}{\hbar}\right)\right)$$

Simplifying the expression, we get:

$$\langle E\rangle = \frac{1}{2}\left(E_a + E_b + A\cos\left(\frac{2At}{\hbar}\right)\right)$$

Now, we can write the Python code to evaluate the expectation value of the energy at time t:

```
        import numpy as np

def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    """
    Evaluate the expectation value of the energy at time t.

    Parameters:
    A (float): Real parameter in the Hamiltonian.
    E_a (float): Energy of level a.
    E_b (float): Energy of level b.
    t (float): Time.

    Returns:
    float: Expectation value of the energy at time t.
    """
    return 0.5 * (E_a + E_b + A * np.cos(2 * A * t / np.hbar))

Example usage:
A = 1.0  # Real parameter in the Hamiltonian
E_a = 2.0  # Energy of level a
E_b = 3.0  # Energy of level b
t = 1.0  # Time

expectation_value_result = expectation_value(A, E_a, E_b, t)
print("Expectation value of the energy at time t:", expectation_value_result)
```

Note that we have used the np.hbar constant from the NumPy library to represent the reduced Planck constant.
Verification Results:

Auto verification result (['2025-02-18T05:50:02.536343+00:00']) Status: **Incorrect**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:58:09.965346Z) Grade: C
Notes: The submitted solution attempts to solve the problem by finding the time-evolution of the state using the Schrdinger equation and calculating the expectation value of the energy. However, there are significant issues: 1) The eigenvalues are incorrectly calculated as $E = \frac{E_a \pm A}{2}$ instead of $E_a \pm A$. 2) The eigenvectors are not normalized, which affects the coefficients c_1, c_2, and c_3. 3) The final expression for the expectation value of energy includes a time-dependent term $A\cos\left(\frac{2At}{\hbar}\right)$, which contradicts the standard solution's result that the expectation value is constant. The approach shows some understanding of quantum mechanics concepts but contains major computational errors.

Attempt 3 (2025-02-01T00:35:04.456047Z)

Detailed Solution: Step 1: Understand the Problem We are given a three-level system with a Hamiltonian H and an initial state $\psi(t=0)$. The task is to find the expectation value of the energy at time t.
Step 2: Devise a Plan To find the expectation value of the energy, we need to calculate the expectation value of the Hamiltonian. This involves finding the time-evolved state $\psi(t)$ and then computing the expectation value of H in this state.
Step 3: Carry Out the Plan First, we need to find the time-evolved state $\psi(t)$. The time-evolution of the state is given by the Schrdinger equation:

$$i\hbar\frac{d\psi(t)}{dt} = H\psi(t)$$

We can solve this differential equation using the given initial condition $\psi(t=0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$.
Step 4: Solve the Time-Evolution Equation The Hamiltonian is given as $H = \begin{pmatrix}E_a & 0 & A\\0 & E_b & 0\\A & 0 & E_a\end{pmatrix}$. We can rewrite the Schrdinger equation as:

$$i\hbar\frac{d\psi(t)}{dt} = \begin{pmatrix}E_a & 0 & A\\0 & E_b & 0\\A & 0 & E_a\end{pmatrix}\psi(t)$$

To solve this equation, we can use the fact that the Hamiltonian is time-independent. The solution to the Schrdinger equation is given by:

$$\psi(t) = e^{-iHt/\hbar}\psi(0)$$

We can compute the time-evolution operator $e^{-iHt/\hbar}$ using the given Hamiltonian.
Step 5: Compute the Time-Evolution Operator The time-evolution operator is given by:

$$e^{-iHt/\hbar} = e^{-i\begin{pmatrix}E_a & 0 & A\\0 & E_b & 0\\A & 0 & E_a\end{pmatrix}t/\hbar}$$

We can compute this exponential using the following formula:

$$e^{AB} = \sum_{n=0}^{\infty}\frac{(AB)^n}{n!}$$

Applying this formula to our Hamiltonian, we get:

$$e^{-iHt/\hbar} = \sum_{n=0}^{\infty}\frac{(-iHt/\hbar)^n}{n!}$$

Step 6: Simplify the Time-Evolution Operator We can simplify the time-evolution operator by computing the powers of the Hamiltonian:

$$H^2 = \begin{pmatrix}E_a & 0 & A\\0 & E_b & 0\\A & 0 & E_a\end{pmatrix}\begin{pmatrix}E_a & 0 & A\\0 & E_b & 0\\A & 0 & E_a\end{pmatrix} = \begin{pmatrix}E_a^2+A^2 & 0 & AE_a\\0 & E_b^2 & 0\\AE_a & 0 & E_a^2+A^2\end{pmatrix}$$

Using this result, we can compute the higher powers of the Hamiltonian.
Step 7: Compute the Expectation Value The expectation value of the energy is given by:

$$\langle E\rangle = \langle\psi(t)|H|\psi(t)\rangle$$

We can compute this expectation value using the time-evolved state $\psi(t)$ and the Hamiltonian H.
Step 8: Simplify the Expectation Value Using the properties of the inner product, we can simplify the expectation value:

$$\langle E\rangle = \psi^\dagger(t)H\psi(t)$$

We can compute this expectation value using the time-evolved state $\psi(t)$ and the Hamiltonian H.
Step 9: Substitute the Time-Evolved State We can substitute the time-evolved state $\psi(t)$ into the expectation value:

$$\langle E \rangle = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} e^{-iE_a t/\hbar} & e^{-iE_b t/\hbar} & e^{-i(E_a t + 2iA^2 t/\hbar + iA^2 t/\hbar)} \end{pmatrix} \right) \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix} \left(\frac{1}{\sqrt{2}} \begin{pmatrix} e^{iE_a t/\hbar} \\ e^{iE_b t/\hbar} \\ e^{i(E_a t + 2iA^2 t/\hbar + iA^2 t/\hbar)} \end{pmatrix} \right)$$

Step 10: Simplify the Expectation Value We can simplify the expectation value by computing the matrix product:

$$\langle E \rangle = \frac{1}{2} \left(E_a + E_b + 2A^2 \cos\left(\frac{2A^2 t}{\hbar} \right) \right)$$

Step 11: Convert to Python Code We can convert the final expression to Python code:

```
        import numpy as np

def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    """
    Compute the expectation value of the energy at time t.

    Parameters:
    A (float): The coupling constant.
    E_a (float): The energy of level a.
    E_b (float): The energy of level b.
    t (float): The time.

    Returns:
    float: The expectation value of the energy at time t.
    """
    return 0.5 * (E_a + E_b + 2 * A**2 * np.cos(2 * A**2 * t / np.hbar))
```

Note that we have used the numpy library to compute the cosine function.

Final Answer: The final answer is $\boxed{\frac{1}{2}(E_a + E_b + 2A^2 \cos(\frac{2A^2 t}{\hbar}))}$.

Verification Results:

Auto verification result (['2025-02-18T05:50:02.536667+00:00']) Status: **Incorrect**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:58:16.743308Z) Grade: C
Notes: The submitted solution shows a basic understanding of quantum mechanics and the process of finding the expectation value of energy. However, it deviates significantly from the standard solution. The standard solution finds the expectation value of energy to be constant, $\langle E \rangle = \frac{1}{2}(E_a + E_b)$, while the submitted solution introduces a time-dependent term $2A^2 \cos\left(\frac{2A^2 t}{\hbar}\right)$, which is incorrect. The approach of solving the time-evolution operator and computing the expectation value is valid, but the final expression does not match the standard solution. The error likely arises from incorrect handling of the time-evolution operator or the initial state decomposition.

Attempt 4 (2025-02-01T00:35:40.140175Z)

Detailed Solution: Step 1: Understand the Problem The problem involves finding the expectation value of the energy of a three-level system at time t. The Hamiltonian of the system is given as a 3×3 matrix, and the initial state of the system is provided.
Step 2: Devise a Plan To find the expectation value of the energy, we need to first find the time-evolved state of the system using the Schrdinger equation. Then, we can calculate the expectation value of the energy using

the formula $\langle E \rangle = \langle \psi(t)|H|\psi(t)\rangle$.
Step 3: Carry Out the Plan First, we need to find the time-evolved state of the system. The time-evolved state is given by the solution to the Schrdinger equation:

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

We can solve this differential equation using the given Hamiltonian and initial state.
Step 4: Solve the Schrdinger Equation The Hamiltonian is given as:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

The initial state is:

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We can write the time-evolved state as:

$$|\psi(t)\rangle = \begin{pmatrix} a(t) \\ b(t) \\ c(t) \end{pmatrix}$$

Substituting this into the Schrdinger equation, we get:

$$i\hbar \frac{d}{dt} \begin{pmatrix} a(t) \\ b(t) \\ c(t) \end{pmatrix} = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix} \begin{pmatrix} a(t) \\ b(t) \\ c(t) \end{pmatrix}$$

This gives us a system of differential equations:

$$i\hbar \frac{da}{dt} = E_a a(t) + Ac(t)$$
$$i\hbar \frac{db}{dt} = E_b b(t)$$
$$i\hbar \frac{dc}{dt} = Aa(t) + E_a c(t)$$

Step 5: Solve the System of Differential Equations We can solve this system of differential equations using standard methods. The solution is:

$$a(t) = \frac{1}{\sqrt{2}} \left(e^{-iE_a t/\hbar} + e^{-iE_b t/\hbar} \right)$$
$$b(t) = e^{-iE_b t/\hbar}$$
$$c(t) = \frac{1}{\sqrt{2}} \left(e^{-iE_a t/\hbar} - e^{-iE_b t/\hbar} \right)$$

Step 6: Calculate the Expectation Value of the Energy The expectation value of the energy is given by:

$$\langle E \rangle = \langle \psi(t)|H|\psi(t)\rangle$$

Substituting the time-evolved state and the Hamiltonian, we get:

$$\langle E \rangle = \left(\frac{1}{\sqrt{2}} \left(e^{-iE_a t/\hbar} + e^{-iE_b t/\hbar} \right), e^{-iE_b t/\hbar}, \frac{1}{\sqrt{2}} \left(e^{-iE_a t/\hbar} - e^{-iE_b t/\hbar} \right) \right) \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \left(e^{-iE_a t/\hbar} + e^{-iE_b t/\hbar} \right) \\ e^{-iE_b t/\hbar} \\ \frac{1}{\sqrt{2}} \left(e^{-iE_a t/\hbar} - e^{-iE_b t/\hbar} \right) \end{pmatrix}$$

Simplifying this expression, we get:

$$\langle E \rangle = \frac{1}{2} (E_a + E_b) + \frac{1}{2} A \left(e^{i(E_a - E_b)t/\hbar} + e^{-i(E_a - E_b)t/\hbar} \right)$$

Step 7: Convert the Final Mathematical Expression into Python Code

```python
import numpy as np

def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    """
    Calculate the expectation value of the energy of a three-level system.

    Parameters:
    A (float): The coupling constant.
    E_a (float): The energy of level a.
    E_b (float): The energy of level b.
    t (float): The time.

    Returns:
    float: The expectation value of the energy.
    """
    # Calculate the expectation value of the energy
    expectation_value = 0.5 * (E_a + E_b) + 0.5 * A * (np.exp(1j * (E_a - E_b) * t /
                                              np.hbar) + np.exp(-1j * (E_a - E_b) *
                                              t / np.hbar))

    return expectation_value
```

Final Answer: The final answer is $\boxed{\frac{1}{2}(E_a + E_b) + \frac{1}{2}A\left(e^{i(E_a - E_b)t/\hbar} + e^{-i(E_a - E_b)t/\hbar}\right)}$.

Verification Results:

Auto verification result (['2025-02-18T05:50:02.537016+00:00']) Status: **Incorrect**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:58:23.296360Z) Grade: C

Notes: The submitted solution shows a basic understanding of the problem and correctly sets up the Schrdinger equation and the Hamiltonian matrix. However, it diverges significantly from the standard solution in the final expression for the expectation value of the energy. The standard solution finds a time-independent expectation value, $\langle E \rangle = \frac{1}{2}(E_a + E_b)$, while the submitted solution includes a time-dependent term, $\frac{1}{2}A(e^{i(E_a - E_b)t/\hbar} + e^{-i(E_a - E_b)t/\hbar})$, which is incorrect. The approach to solving the differential equations is valid, but the final expression does not match the standard solution, indicating a significant error in the calculation of the expectation value.

Attempt 5 (2025-02-01T00:36:20.725030Z)

Detailed Solution: Problem Solution
Step 1: Understand the Problem
We are given a three-level system with a Hamiltonian H and an initial state $\psi(t = 0)$. We need to find the expectation value of the energy at time t.
Step 2: Devise a Plan
To find the expectation value of the energy, we need to first find the time-evolved state $\psi(t)$ using the time-dependent Schrdinger equation. Then, we can calculate the expectation value of the energy using the formula $\langle E \rangle = \langle \psi(t)|H|\psi(t) \rangle$.
Step 3: Carry Out the Plan
The time-evolved state $\psi(t)$ can be found by solving the time-dependent Schrdinger equation:
$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$
We can solve this equation by finding the eigenvalues and eigenvectors of the Hamiltonian H.
Step 4: Find the Eigenvalues and Eigenvectors of H
The Hamiltonian H is given by:

$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$
We can find the eigenvalues and eigenvectors of H by solving the characteristic equation:
$|H - EI| = 0$
where I is the identity matrix.
Step 5: Solve the Characteristic Equation
The characteristic equation is given by:
$\begin{vmatrix} E_a - E & 0 & A \\ 0 & E_b - E & 0 \\ A & 0 & E_a - E \end{vmatrix} = 0$
Expanding the determinant, we get:
$(E_a - E)(E_b - E)(E_a - E) - A^2(E_b - E) = 0$
Simplifying the equation, we get:
$E^3 - (E_a + E_b + E_a)E^2 + (E_aE_b + E_a^2 + E_bE_a)E - E_aE_bE_a - A^2E_b = 0$
Step 6: Find the Eigenvalues
The eigenvalues of H are the solutions to the characteristic equation:
$E^3 - (2E_a + E_b)E^2 + (E_aE_b + 2E_a^2)E - E_aE_bE_a - A^2E_b = 0$
Let the eigenvalues be E_1, E_2, and E_3. We can find the eigenvalues by solving the cubic equation.
Step 7: Find the Eigenvectors
Once we have the eigenvalues, we can find the corresponding eigenvectors by solving the equation:
$H|v_i\rangle = E_i|v_i\rangle$
where $|v_i\rangle$ is the eigenvector corresponding to the eigenvalue E_i.
Step 8: Find the Time-Evolved State
The time-evolved state $\psi(t)$ can be found by expanding it in terms of the eigenstates:
$|\psi(t)\rangle = \sum_{i=1}^{3} c_i e^{-iE_it/\hbar}|v_i\rangle$
where c_i are the coefficients of the expansion.
Step 9: Find the Expectation Value of the Energy
The expectation value of the energy can be found using the formula:
$\langle E\rangle = \langle\psi(t)|H|\psi(t)\rangle$
Substituting the expression for $\psi(t)$, we get:
$\langle E\rangle = \sum_{i=1}^{3}\sum_{j=1}^{3} c_i^* c_j e^{-i(E_i - E_j)t/\hbar}\langle v_i|H|v_j\rangle$
Step 10: Simplify the Expression
Simplifying the expression, we get:
$\langle E\rangle = \sum_{i=1}^{3} |c_i|^2 E_i + \sum_{i\neq j} c_i^* c_j e^{-i(E_i - E_j)t/\hbar}\langle v_i|H|v_j\rangle$
Step 11: Find the Coefficients c_i
The coefficients c_i can be found by expanding the initial state $\psi(0)$ in terms of the eigenstates:
$\psi(0) = \sum_{i=1}^{3} c_i|v_i\rangle$
where c_i are the coefficients of the expansion.
Step 12: Simplify the Expression for $\langle E\rangle$
Simplifying the expression for $\langle E\rangle$, we get:
$\langle E\rangle = \frac{1}{2}(E_a + E_b) + \frac{1}{2}(E_a - E_b + 2A\cos(\omega t))$
where $\omega = \sqrt{(E_a - E_b)^2 + 4A^2}$.
Step 13: Write the Final Expression for $\langle E\rangle$
The final expression for $\langle E\rangle$ is:
$\langle E\rangle = \frac{1}{2}(E_a + E_b) + \frac{1}{2}(E_a - E_b + 2A\cos(\omega t))$
where $\omega = \sqrt{(E_a - E_b)^2 + 4A^2}$.
Step 14: Convert the Final Expression to Python Code

```
        import numpy as np

def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    """
    Calculate the expectation value of the energy at time t.

    Parameters:
    A (float): The coupling constant.
    E_a (float): The energy of level a.
```

```
    E_b (float): The energy of level b.
    t (float): The time.

    Returns:
    float: The expectation value of the energy at time t.
    """
    omega = np.sqrt((E_a - E_b)**2 + 4*A**2)
    return 0.5 * (E_a + E_b) + 0.5 * (E_a - E_b + 2*A * np.cos(omega * t))
```

Final Answer:

The final answer is $\boxed{\frac{1}{2}(E_a + E_b) + \frac{1}{2}\left(E_a - E_b + 2A\cos(\omega t)\right)}$ where $\omega = \sqrt{(E_a - E_b)^2 + 4A^2}$.

Verification Results:

Auto verification result (['2025-02-18T05:50:02.537460+00:00']) Status: **Incorrect**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:58:29.705870Z) Grade: C

Notes: The submitted solution shows a basic understanding of the problem by attempting to find the expectation value of the energy using the time-dependent Schrdinger equation. However, it deviates significantly from the standard solution. The standard solution finds the expectation value of energy as a constant, $\langle E \rangle = \frac{1}{2}(E_a + E_b)$, while the submitted solution introduces a time-dependent term $\cos(\omega t)$, which is not present in the standard solution. This indicates a major flaw in the approach, likely due to incorrect handling of the eigenstates or the time evolution. The submitted solution also includes unnecessary steps and complexity that do not align with the standard solution's approach.

2.2.10 Model: Qwen/Qwen2.5-7B-Instruct

Attempt 1 (2025-02-01T02:49:11.356291Z)

Detailed Solution: To find the expectation value of the energy at time t, we need to follow these steps: 1. **Diagonalize the Hamiltonian** H to find the eigenstates and eigenvalues. 2. **Express the initial state** $\psi(0)$ in terms of the eigenstates of H. 3. **Find the time-evolved state** $\psi(t)$. 4. **Calculate the expectation value** of the energy using the time-evolved state.

Step 1: Diagonalize the Hamiltonian

The Hamiltonian is given by:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

To diagonalize H, we need to find its eigenvalues and eigenvectors. The characteristic equation is:

$$\det(H - \lambda I) = 0$$

This gives:

$$\begin{vmatrix} E_a - \lambda & 0 & A \\ 0 & E_b - \lambda & 0 \\ A & 0 & E_a - \lambda \end{vmatrix} = 0$$

Expanding the determinant, we get:

$$(E_a - \lambda)^2 (E_b - \lambda) - A^2 (E_b - \lambda) = 0$$

$$(E_b - \lambda) \left[(E_a - \lambda)^2 - A^2 \right] = 0$$

This gives us the eigenvalues:

$$\lambda_1 = E_b$$

$$\lambda_2 = E_a + A$$

$$\lambda_3 = E_a - A$$

Step 2: Find the Eigenvectors

For $\lambda_1 = E_b$:

$$\begin{pmatrix} E_a - E_b & 0 & A \\ 0 & E_b - E_b & 0 \\ A & 0 & E_a - E_b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} E_a - E_b & 0 & A \\ 0 & 0 & 0 \\ A & 0 & E_a - E_b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

This gives:

$$(E_a - E_b) x_1 + A x_3 = 0$$

$$A x_1 + (E_a - E_b) x_3 = 0$$

Solving these, we get:

$$x_1 = -\frac{A}{E_a - E_b} x_3$$

$$x_2 = 0$$

So, the eigenvector is:

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{A}{E_a - E_b} \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda_2 = E_a + A$:

$$\begin{pmatrix} E_a - (E_a + A) & 0 & A \\ 0 & E_b - (E_a + A) & 0 \\ A & 0 & E_a - (E_a + A) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} -A & 0 & A \\ 0 & E_b - E_a - A & 0 \\ A & 0 & -A \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

This gives:

$$-Ax_1 + Ax_3 = 0$$

$$(E_b - E_a - A)x_2 = 0$$

Solving these, we get:

$$x_1 = x_3$$

$$x_2 = 0$$

So, the eigenvector is:

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda_3 = E_a - A$:

$$\begin{pmatrix} E_a - (E_a - A) & 0 & A \\ 0 & E_b - (E_a - A) & 0 \\ A & 0 & E_a - (E_a - A) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} A & 0 & A \\ 0 & E_b - E_a + A & 0 \\ A & 0 & A \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

This gives:

$$Ax_1 + Ax_3 = 0$$

$$(E_b - E_a + A)x_2 = 0$$

Solving these, we get:

$$x_1 = -x_3$$

$$x_2 = 0$$

So, the eigenvector is:

$$\mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Step 3: Express the Initial State in Terms of Eigenvectors
The initial state is:

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We express $\psi(0)$ in terms of the eigenvectors:

$$\psi(0) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

Solving for c_1, c_2, c_3:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} -\frac{A}{E_a - E_b} \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

This gives:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{Ac_1}{E_a - E_b} + c_2 - c_3 \\ 0 \\ c_1 + c_2 + c_3 \end{pmatrix}$$

Solving these, we get:

$$-\frac{Ac_1}{E_a - E_b} + c_2 - c_3 = \frac{1}{\sqrt{2}}$$

$$c_1 + c_2 + c_3 = 0$$

$$c_2 + c_3 = \frac{1}{\sqrt{2}}$$

Solving these equations, we find:

$$c_1 = \frac{1}{\sqrt{2}}\left(\frac{1}{E_a - E_b} - 1\right)$$
$$c_2 = \frac{1}{\sqrt{2}}\left(\frac{1}{E_a - E_b} + 1\right)$$
$$c_3 = 0$$

Step 4: Time-Evolved State
The time-evolved state is:

$$\psi(t) = c_1 e^{-i\lambda_1 t/\hbar}\mathbf{v}_1 + c_2 e^{-i\lambda_2 t/\hbar}\mathbf{v}_2 + c_3 e^{-i\lambda_3 t/\hbar}\mathbf{v}_3$$

The expectation value of the energy is:

$$\langle H\rangle = \sum_i c_i^* c_i \lambda_i$$

Substituting the values:

$$\langle H\rangle = |c_1|^2 \lambda_1 + |c_2|^2 \lambda_2 + |c_3|^2 \lambda_3$$

Finally, we get:

$$\langle H\rangle = \left|\frac{1}{\sqrt{2}}\left(\frac{1}{E_a - E_b} - 1\right)\right|^2 E_b + \left|\frac{1}{\sqrt{2}}\left(\frac{1}{E_a - E_b} + 1\right)\right|^2 (E_a + A)$$

Simplifying:

$$\langle H\rangle = \frac{1}{2}\left(\frac{1}{E_a - E_b} - 1\right)^2 E_b + \frac{1}{2}\left(\frac{1}{E_a - E_b} + 1\right)^2 (E_a + A)$$

Final Answer:

$$\boxed{\langle H\rangle = \frac{1}{2}\left(\frac{1}{E_a - E_b} - 1\right)^2 E_b + \frac{1}{2}\left(\frac{1}{E_a - E_b} + 1\right)^2 (E_a + A)}$$

Python Code:

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    term1 = (1 / (E_a - E_b) - 1) ** 2 * E_b
    term2 = (1 / (E_a - E_b) + 1) ** 2 * (E_a + A)
    return 0.5 * term1 + 0.5 * term2
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.537862+00:00']) Status: **Incorrect**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:58:37.355589Z) Grade: D
Notes: The submitted solution has significant issues. The eigenvectors and eigenvalues are incorrectly calculated, leading to an incorrect expression for the expectation value of the energy. The standard solution correctly identifies the eigenstates as $\frac{1}{\sqrt{2}}(1, 0, \pm 1)^T$ and $(0, 1, 0)^T$ with energies $E_a \pm A$ and E_b. The submitted solution's final expression for $\langle H\rangle$ is not equivalent to the standard solution's $\langle E\rangle = \frac{1}{2}(E_a + E_b)$. The approach to finding the expectation value is flawed, and the final answer is incorrect. The Python code provided also reflects this incorrect approach.

Attempt 2 (2025-02-01T02:49:37.373546Z)

Detailed Solution: To find the expectation value of the energy at time t, we need to follow these steps:
1. **Diagonalize the Hamiltonian** H. 2. **Find the time evolution of the state** $\psi(t)$. 3. **Calculate the expectation value** $\langle H\rangle$ at time t.
Step 1: Diagonalize the Hamiltonian
The Hamiltonian is given by:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

To diagonalize H, we need to find its eigenvalues and eigenvectors. The characteristic equation is:

$$\det(H - \lambda I) = 0$$

This gives:

$$\begin{vmatrix} E_a - \lambda & 0 & A \\ 0 & E_b - \lambda & 0 \\ A & 0 & E_a - \lambda \end{vmatrix} = 0$$

Expanding the determinant, we get:

$$(E_a - \lambda)^2 (E_b - \lambda) - A^2 (E_b - \lambda) = 0$$

$$(E_b - \lambda) \left[(E_a - \lambda)^2 - A^2 \right] = 0$$

This gives us the eigenvalues:

$$\lambda_1 = E_b$$

$$\lambda_2 = E_a + A$$

$$\lambda_3 = E_a - A$$

Step 2: Find the Time Evolution of the State
The state at $t = 0$ is:

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

The time evolution of the state is given by:

$$\psi(t) = U(t)\psi(0)$$

where $U(t) = e^{-iHt/\hbar}$.
Since H is diagonalizable, we can write:

$$U(t) = V e^{-i\Lambda t/\hbar} V^{-1}$$

where V is the matrix of eigenvectors and Λ is the diagonal matrix of eigenvalues.
Step 3: Calculate the Expectation Value
The expectation value of the energy at time t is:

$$\langle H \rangle(t) = \langle \psi(t) | H | \psi(t) \rangle$$

Using the time evolution, we have:

$$\langle H \rangle(t) = \langle \psi(0) | U^\dagger(t) H U(t) | \psi(0) \rangle$$

Since $U(t)$ is unitary, $U^\dagger(t) = U^{-1}(t)$, and we can write:

$$\langle H \rangle(t) = \langle \psi(0) | \Lambda | \psi(0) \rangle$$

Given the state $\psi(0)$, we can compute the expectation value directly:

$$\langle H \rangle(t) = \frac{1}{2} \left[\langle 1 | \lambda_1 | 1 \rangle + \langle 1 | \lambda_2 | 1 \rangle + \langle 1 | \lambda_3 | 1 \rangle \right]$$

Since $\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, we have:

$$\langle H \rangle(t) = \frac{1}{2} \left[\frac{1}{\sqrt{2}} (\lambda_1 + \lambda_2) \right] = \frac{1}{2} \left[\frac{1}{\sqrt{2}} (E_b + (E_a + A)) \right]$$

Simplifying, we get:

$$\langle H \rangle(t) = \frac{1}{2} \left(\frac{E_b + E_a + A}{\sqrt{2}} \right) = \frac{E_b + E_a + A}{2\sqrt{2}}$$

Final Answer: The expectation value of the energy at time t is:

$$\boxed{\frac{E_b + E_a + A}{2\sqrt{2}}}$$

Python Code:

```
def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    return (E_b + E_a + A) / (2 * (2 ** 0.5))
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.538225+00:00']) Status: **Incorrect**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:58:43.688755Z) Grade: D

Notes: The submitted solution is fundamentally flawed. The standard solution shows that the expectation value of the energy is constant and equal to $\frac{1}{2}(E_a+E_b)$. However, the submitted solution incorrectly calculates the expectation value as $\frac{E_b+E_a+A}{2\sqrt{2}}$, which is not equivalent to the standard solution. The approach taken in the submitted solution is incorrect, as it misapplies the concept of time evolution and expectation value calculation. The eigenstates and their contributions to the expectation value are not correctly considered, leading to a wrong final answer.

Attempt 3 (2025-02-01T02:50:30.864210Z)

Detailed Solution: To find the expectation value of the energy at time t, we need to follow these steps: 1. **Diagonalize the Hamiltonian** to find the eigenvalues and eigenvectors. 2. **Express the initial state** in the eigenbasis of the Hamiltonian. 3. **Calculate the time evolution** of the state. 4. **Compute the expectation value** of the energy at time t.

Step 1: Diagonalize the Hamiltonian

The Hamiltonian is given by:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

To diagonalize H, we need to find its eigenvalues and eigenvectors. The characteristic equation is:

$$\det(H - \lambda I) = 0$$

$$\det \begin{pmatrix} E_a - \lambda & 0 & A \\ 0 & E_b - \lambda & 0 \\ A & 0 & E_a - \lambda \end{pmatrix} = 0$$

This determinant can be expanded as:

$$(E_a - \lambda)^2 (E_b - \lambda) - A^2 (E_b - \lambda) = 0$$

$$(E_b - \lambda) \left[(E_a - \lambda)^2 - A^2 \right] = 0$$

So, the eigenvalues are:

$$\lambda_1 = E_b$$

$$\lambda_2 = E_a + A$$

$$\lambda_3 = E_a - A$$

Step 2: Find the Eigenvectors

For $\lambda_1 = E_b$:

$$\begin{pmatrix} E_a - E_b & 0 & A \\ 0 & E_b - E_b & 0 \\ A & 0 & E_a - E_b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} E_a - E_b & 0 & A \\ 0 & 0 & 0 \\ A & 0 & E_a - E_b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

This gives us:

$$(E_a - E_b) x_1 + A x_3 = 0$$

$$Ax_1 + (E_a - E_b)x_3 = 0$$

Solving these, we get:

$$x_1 = -\frac{A}{E_a - E_b}x_3$$
$$x_2 = 0$$

So, the eigenvector is:

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{A}{E_a - E_b} \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda_2 = E_a + A$:

$$\begin{pmatrix} E_a - (E_a + A) & 0 & A \\ 0 & E_b - (E_a + A) & 0 \\ A & 0 & E_a - (E_a + A) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} -A & 0 & A \\ 0 & E_b - E_a - A & 0 \\ A & 0 & -A \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

This gives us:

$$-Ax_1 + Ax_3 = 0$$
$$(E_b - E_a - A)x_2 = 0$$

Solving these, we get:

$$x_1 = x_3$$
$$x_2 = 0$$

So, the eigenvector is:

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda_3 = E_a - A$:

$$\begin{pmatrix} E_a - (E_a - A) & 0 & A \\ 0 & E_b - (E_a - A) & 0 \\ A & 0 & E_a - (E_a - A) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} A & 0 & A \\ 0 & E_b - E_a + A & 0 \\ A & 0 & A \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

This gives us:

$$Ax_1 + Ax_3 = 0$$
$$(E_b - E_a + A)x_2 = 0$$

Solving these, we get:

$$x_1 = -x_3$$
$$x_2 = 0$$

So, the eigenvector is:

$$\mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Step 3: Express the Initial State in the Eigenbasis
The initial state is:

$$\psi(t = 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We express this in the eigenbasis:

$$\psi(t = 0) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

Solving for c_1, c_2, c_3:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} = c_1\begin{pmatrix}-\frac{A}{E_a-E_b}\\0\\1\end{pmatrix} + c_2\begin{pmatrix}1\\0\\1\end{pmatrix} + c_3\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

This gives us:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}-\frac{A}{E_a-E_b}c_1 + c_2 - c_3\\0\\c_1 + c_2 + c_3\end{pmatrix}$$

From the third component:

$$c_1 + c_2 + c_3 = 0$$

From the first component:

$$-\frac{A}{E_a - E_b}c_1 + c_2 - c_3 = \frac{1}{\sqrt{2}}$$

From the second component:

$$0 = \frac{1}{\sqrt{2}}$$

Solving these, we get:

$$c_2 = \frac{1}{\sqrt{2}}$$

$$c_1 + \frac{1}{\sqrt{2}} + c_3 = 0$$

$$-\frac{A}{E_a - E_b}c_1 + \frac{1}{\sqrt{2}} - c_3 = \frac{1}{\sqrt{2}}$$

Solving these, we get:

$$c_1 = \frac{1}{\sqrt{2}}$$

$$c_3 = -\frac{1}{\sqrt{2}}$$

So, the initial state in the eigenbasis is:

$$\psi(t=0) = \frac{1}{\sqrt{2}}\mathbf{v}_2 - \frac{1}{\sqrt{2}}\mathbf{v}_3$$

Step 4: Time Evolution
The time evolution of the state is:

$$\psi(t) = e^{-iHt/\hbar}\psi(t=0)$$

Since the Hamiltonian is diagonal in the eigenbasis, the time evolution is:

$$\psi(t) = e^{-i\lambda_2 t/\hbar}\mathbf{v}_2 - e^{-i\lambda_3 t/\hbar}\mathbf{v}_3$$

Step 5: Expectation Value of Energy
The expectation value of the energy is:

$$\langle H\rangle = \sum_i c_i^* \lambda_i c_i$$

Substituting the values:

$$\langle H\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 (E_a + A) + \left(-\frac{1}{\sqrt{2}}\right)^2 (E_a - A)$$

$$\langle H\rangle = \frac{1}{2}(E_a + A) + \frac{1}{2}(E_a - A)$$

$$\langle H\rangle = E_a$$

Thus, the expectation value of the energy at time t is:

$$\boxed{E_a}$$

Python Code

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    return E_a
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.538564+00:00']) Status: **Incorrect**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:58:51.116568Z) Grade: C

Notes: The submitted solution shows a basic understanding of the problem by attempting to diagonalize the Hamiltonian and find the eigenvalues and eigenvectors. However, there are significant issues. The final expectation value of energy is incorrectly calculated as E_a, whereas the standard solution shows it should be $\frac{1}{2}(E_a + E_b)$. The submitted solution incorrectly assumes the initial state and its decomposition in the eigenbasis, leading to an incorrect final result. The approach has merit, but the execution contains major flaws that lead to the wrong conclusion.

Attempt 4 (2025-02-01T02:51:11.441004Z)

Detailed Solution: To find the expectation value of the energy at time t, we need to follow these steps: 1. **Diagonalize the Hamiltonian**: Find the eigenvalues and eigenvectors of the Hamiltonian H. 2. **Express the initial state in the eigenbasis**: Write the initial state $\psi(t = 0)$ in terms of the eigenstates of H. 3. **Calculate the time evolution of the state**: Use the time evolution operator to find the state at time t. 4. **Compute the expectation value of the energy**: Use the time-evolved state to find the expectation value of the Hamiltonian.

Step 1: Diagonalize the Hamiltonian

The Hamiltonian is given by:

$$H = \begin{pmatrix} E_a & 0 & A \\ 0 & E_b & 0 \\ A & 0 & E_a \end{pmatrix}$$

To find the eigenvalues, we solve the characteristic equation $\det(H - \lambda I) = 0$:

$$\det \begin{pmatrix} E_a - \lambda & 0 & A \\ 0 & E_b - \lambda & 0 \\ A & 0 & E_a - \lambda \end{pmatrix} = 0$$

This simplifies to:

$$(E_a - \lambda)^2 (E_b - \lambda) - A^2 (E_b - \lambda) = 0$$

$$(E_b - \lambda) \left[(E_a - \lambda)^2 - A^2 \right] = 0$$

The eigenvalues are:

$$\lambda_1 = E_b$$

$$\lambda_2 = E_a + A$$

$$\lambda_3 = E_a - A$$

Step 2: Find the Eigenvectors

For $\lambda_1 = E_b$:

$$\begin{pmatrix} E_a - E_b & 0 & A \\ 0 & E_b - E_b & 0 \\ A & 0 & E_a - E_b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

This gives:

$$(E_a - E_b)x_1 + Ax_3 = 0$$

$$Ax_1 = 0$$

$$(E_a - E_b)x_3 = 0$$

The eigenvector is:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda_2 = E_a + A$:

$$\begin{pmatrix} E_a - (E_a + A) & 0 & A \\ 0 & E_b - (E_a + A) & 0 \\ A & 0 & E_a - (E_a + A) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

This gives:

$$-Ax_1 + Ax_3 = 0$$

$$(E_b - E_a - A)x_2 = 0$$

$$-Ax_1 + (E_a - E_a - A)x_3 = 0$$

The eigenvector is:

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda_3 = E_a - A$:

$$\begin{pmatrix} E_a - (E_a - A) & 0 & A \\ 0 & E_b - (E_a - A) & 0 \\ A & 0 & E_a - (E_a - A) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

This gives:

$$Ax_1 + Ax_3 = 0$$

$$(E_b - E_a + A)x_2 = 0$$

$$Ax_1 + Ax_3 = 0$$

The eigenvector is:

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Step 3: Express the Initial State in the Eigenbasis
The initial state is:

$$\psi(t = 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We express this in terms of the eigenstates:

$$\psi(t = 0) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

Solving for c_1, c_2, c_3:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

This gives:

$$c_1 + c_2 + c_3 = \frac{1}{\sqrt{2}}$$

$$c_2 - c_3 = \frac{1}{\sqrt{2}}$$

Solving these equations, we get:

$$c_1 = 0, \quad c_2 = \frac{1}{\sqrt{2}}, \quad c_3 = 0$$

Step 4: Compute the Expectation Value of the Energy
The time-evolved state is:

$$\psi(t) = e^{-iHt/\hbar}\psi(t = 0)$$

Since $\psi(t = 0)$ is already in the eigenbasis, the time-evolved state is:

$$\psi(t) = \frac{1}{\sqrt{2}}\mathbf{v}_2 e^{-i(E_a+A)t/\hbar}$$

The expectation value of the energy is:

$$\langle H \rangle = \psi(t)^\dagger H \psi(t)$$

Substituting the time-evolved state:

$$\langle H \rangle = \left(\frac{1}{\sqrt{2}} \mathbf{v}_2^\dagger e^{i(E_a + A)t/\hbar} \right) H \left(\frac{1}{\sqrt{2}} \mathbf{v}_2 e^{-i(E_a + A)t/\hbar} \right)$$

$$\langle H \rangle = \frac{1}{2} \mathbf{v}_2^\dagger H \mathbf{v}_2$$

Since $\mathbf{v}_2$ is an eigenvector of H with eigenvalue $E_a + A$:

$$\langle H \rangle = \frac{1}{2}(E_a + A)$$

Final Answer: The expectation value of the energy at time t is:

$$\boxed{\frac{1}{2}(E_a + A)}$$

Python Code:

```
        def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    return 0.5 * (E_a + A)
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.538908+00:00']) Status: **Incorrect**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:58:58.427070Z) Grade: C

Notes: The submitted solution has significant issues. The eigenvectors and eigenvalues are correctly calculated, but the final expectation value of the energy is incorrect. The standard solution shows that the expectation value should be $\langle E \rangle = \frac{1}{2}(E_a + E_b)$, while the submitted solution incorrectly concludes $\frac{1}{2}(E_a + A)$. The approach to find the eigenvectors and eigenvalues is valid, but the decomposition of the initial state and the calculation of the expectation value are flawed. The submitted solution does not correctly express the initial state in terms of the eigenbasis, leading to an incorrect final result.

Attempt 5 (2025-02-01T02:51:53.521943Z)

Detailed Solution: To find the expectation value of the energy at time t, we need to follow these steps: 1. **Find the time-evolved state vector** $\psi(t)$. 2. **Calculate the expectation value** $\langle H \rangle(t)$ of the Hamiltonian H with respect to $\psi(t)$.

Step 1: Time-Evolved State Vector

The time-evolved state vector $\psi(t)$ is given by:

$$\psi(t) = e^{-iHt/\hbar} \psi(0)$$

Since $\hbar = 1$ for simplicity, we have:

$$\psi(t) = e^{-iHt} \psi(0)$$

First, we need to diagonalize the Hamiltonian H. The eigenvalues and eigenvectors of H are required.

Diagonalizing the Hamiltonian

The characteristic equation of H is:

$$\det(H - \lambda I) = 0$$

$$\begin{vmatrix} E_a - \lambda & 0 & A \\ 0 & E_b - \lambda & 0 \\ A & 0 & E_a - \lambda \end{vmatrix} = 0$$

Expanding the determinant, we get:

$$(E_a - \lambda)^2 (E_b - \lambda) - A^2 (E_b - \lambda) = 0$$

$$(E_b - \lambda)\left[(E_a - \lambda)^2 - A^2\right] = 0$$

This gives us the eigenvalues:

$$\lambda_1 = E_b$$

$$(E_a - \lambda)^2 - A^2 = 0 \implies \lambda_2, \lambda_3 = E_a \pm A$$

The eigenvalues are:

$$\lambda_1 = E_b, \quad \lambda_2 = E_a + A, \quad \lambda_3 = E_a - A$$

Next, we find the eigenvectors corresponding to these eigenvalues.
Eigenvectors
For $\lambda_1 = E_b$:

$$\begin{pmatrix} E_a - E_b & 0 & A \\ 0 & E_b - E_b & 0 \\ A & 0 & E_a - E_b \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$(E_a - E_b)x_1 + Ax_3 = 0$$

$$Ax_1 = 0 \implies x_1 = 0$$

$$Ax_3 = 0 \implies x_3 = 0$$

$$x_2 \text{ is free}$$

Thus, the eigenvector is:

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_2 = E_a + A$:

$$\begin{pmatrix} E_a - (E_a + A) & 0 & A \\ 0 & E_b - (E_a + A) & 0 \\ A & 0 & E_a - (E_a + A) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-Ax_1 + Ax_3 = 0 \implies x_1 = x_3$$

$$(E_b - E_a - A)x_2 = 0 \implies x_2 \text{ is free}$$

Thus, the eigenvector is:

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda_3 = E_a - A$:

$$\begin{pmatrix} E_a - (E_a - A) & 0 & A \\ 0 & E_b - (E_a - A) & 0 \\ A & 0 & E_a - (E_a - A) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$Ax_1 + Ax_3 = 0 \implies x_1 = -x_3$$

$$(E_b - E_a + A)x_2 = 0 \implies x_2 \text{ is free}$$

Thus, the eigenvector is:

$$\mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The matrix U of eigenvectors is:

$$U = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

The diagonal matrix D of eigenvalues is:

$$D = \begin{pmatrix} E_b & 0 & 0 \\ 0 & E_a + A & 0 \\ 0 & 0 & E_a - A \end{pmatrix}$$

The time-evolution operator is:

$$e^{-iHt} = Ue^{-iDt}U^{-1}$$

Step 2: Expectation Value
The expectation value of the energy at time t is:

$$\langle H\rangle(t) = \langle\psi(t)|H|\psi(t)\rangle$$

Since $\psi(t) = Ue^{-iDt}U^{-1}\psi(0)$, we have:

$$\langle H\rangle(t) = \langle\psi(0)|Ue^{iDt}U^{-1}HUe^{-iDt}U^{-1}|\psi(0)\rangle$$

Given $\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$, we can compute:

$$\langle H\rangle(t) = \frac{1}{2}\left(\langle 1|Ue^{iDt}U^{-1}HUe^{-iDt}U^{-1}|1\rangle + \langle 1|Ue^{iDt}U^{-1}HUe^{-iDt}U^{-1}|1\rangle\right)$$

Since $\psi(0)$ is a superposition of $\mathbf{v}_2$ and $\mathbf{v}_3$, we can simplify:

$$\langle H\rangle(t) = \frac{1}{2}\left(\langle 1|Ue^{iDt}U^{-1}HUe^{-iDt}U^{-1}|1\rangle + \langle 1|Ue^{iDt}U^{-1}HUe^{-iDt}U^{-1}|1\rangle\right)$$

Finally, the expectation value is:

$$\langle H\rangle(t) = \frac{1}{2}\left((E_a + A)\cos(At) + (E_a - A)\cos(At)\right) = (E_a)\cos(At)$$

Final Answer:

$$\boxed{(E_a)\cos(At)}$$

Python Code:

```
        import numpy as np

def expectation_value(A: float, E_a: float, E_b: float, t: float) -> float:
    return E_a * np.cos(A * t)
```

Verification Results:

Auto verification result (['2025-02-18T05:50:02.539271+00:00']) Status: **Incorrect**

Evaluations:

Evaluated by gpt-4o (2025-02-04T02:59:05.665197Z) Grade: D
Notes: The submitted solution is fundamentally flawed. The standard solution shows that the expectation value of the energy is constant and given by $\langle E\rangle = \frac{1}{2}(E_a + E_b)$. However, the submitted solution incorrectly calculates a time-dependent expectation value $(E_a)\cos(At)$, which is not consistent with the standard solution. The approach taken in the submitted solution involves unnecessary diagonalization and time evolution, which are not required for finding the expectation value of a time-independent Hamiltonian in the given state. The final answer is incorrect and does not match the standard solution.